Introduction to Probability
2nd Edition
Problem Solutions

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CHAPTER 1

Solution to Problem 1.1. We have

$$A = \{2,4,6\}, \qquad B = \{4,5,6\},$$

so $A \cup B = \{2,4,5,6\}$, and

$$(A \cup B)^c = \{1,3\}.$$

On the other hand,

$$A^c \cap B^c = \{1,3,5\} \cap \{1,2,3\} = \{1,3\}.$$

Similarly, we have $A \cap B = \{4,6\}$, and

$$(A \cap B)^c = \{1,2,3,5\}.$$

On the other hand,

$$A^c \cup B^c = \{1,3,5\} \cup \{1,2,3\} = \{1,2,3,5\}.$$

Solution to Problem 1.2. (a) By using a Venn diagram it can be seen that for any sets S and T, we have

$$S = (S \cap T) \cup (S \cap T^c).$$

(Alternatively, argue that any x must belong to either T or to T^c, so x belongs to S if and only if it belongs to $S \cap T$ or to $S \cap T^c$.) Apply this equality with $S = A^c$ and $T = B$, to obtain the first relation

$$A^c = (A^c \cap B) \cup (A^c \cap B^c).$$

Interchange the roles of A and B to obtain the second relation.

(b) By De Morgan's law, we have

$$(A \cap B)^c = A^c \cup B^c,$$

and by using the equalities of part (a), we obtain

$$(A \cap B)^c = \big((A^c \cap B) \cup (A^c \cap B^c)\big) \cup \big((A \cap B^c) \cup (A^c \cap B^c)\big) = (A^c \cap B) \cup (A^c \cap B^c) \cup (A \cap B^c).$$

(c) We have $A = \{1,3,5\}$ and $B = \{1,2,3\}$, so $A \cap B = \{1,3\}$. Therefore,

$$(A \cap B)^c = \{2,4,5,6\},$$

and

$$A^c \cap B = \{2\}, \qquad A^c \cap B^c = \{4, 6\}, \qquad A \cap B^c = \{5\}.$$

Thus, the equality of part (b) is verified.

Solution to Problem 1.5. Let G and C be the events that the chosen student is a genius and a chocolate lover, respectively. We have $\mathbf{P}(G) = 0.6$, $\mathbf{P}(C) = 0.7$, and $\mathbf{P}(G \cap C) = 0.4$. We are interested in $\mathbf{P}(G^c \cap C^c)$, which is obtained with the following calculation:

$$\mathbf{P}(G^c \cap C^c) = 1 - \mathbf{P}(G \cup C) = 1 - \big(\mathbf{P}(G) + \mathbf{P}(C) - \mathbf{P}(G \cap C)\big) = 1 - (0.6 + 0.7 - 0.4) = 0.1.$$

Solution to Problem 1.6. We first determine the probabilities of the six possible outcomes. Let $a = \mathbf{P}(\{1\}) = \mathbf{P}(\{3\}) = \mathbf{P}(\{5\})$ and $b = \mathbf{P}(\{2\}) = \mathbf{P}(\{4\}) = \mathbf{P}(\{6\})$. We are given that $b = 2a$. By the additivity and normalization axioms, $1 = 3a + 3b = 3a + 6a = 9a$. Thus, $a = 1/9$, $b = 2/9$, and $\mathbf{P}(\{1, 2, 3\}) = 4/9$.

Solution to Problem 1.7. The outcome of this experiment can be any finite sequence of the form $(a_1, a_2, \ldots, a_n)$, where n is an arbitrary positive integer, $a_1, a_2, \ldots, a_{n-1}$ belong to $\{1, 3\}$, and a_n belongs to $\{2, 4\}$. In addition, there are possible outcomes in which an even number is never obtained. Such outcomes are infinite sequences $(a_1, a_2, \ldots)$, with each element in the sequence belonging to $\{1, 3\}$. The sample space consists of all possible outcomes of the above two types.

Solution to Problem 1.8. Let p_i be the probability of winning against the opponent played in the ith turn. Then, you will win the tournament if you win against the 2nd player (probability p_2) and also you win against at least one of the two other players [probability $p_1 + (1 - p_1)p_3 = p_1 + p_3 - p_1 p_3$]. Thus, the probability of winning the tournament is

$$p_2(p_1 + p_3 - p_1 p_3).$$

The order $(1, 2, 3)$ is optimal if and only if the above probability is no less than the probabilities corresponding to the two alternative orders, i.e.,

$$p_2(p_1 + p_3 - p_1 p_3) \geq p_1(p_2 + p_3 - p_2 p_3),$$

$$p_2(p_1 + p_3 - p_1 p_3) \geq p_3(p_2 + p_1 - p_2 p_1).$$

It can be seen that the first inequality above is equivalent to $p_2 \geq p_1$, while the second inequality above is equivalent to $p_2 \geq p_3$.

Solution to Problem 1.9. (a) Since $\Omega = \cup_{i=1}^n S_i$, we have

$$A = \bigcup_{i=1}^{n} (A \cap S_i),$$

while the sets $A \cap S_i$ are disjoint. The result follows by using the additivity axiom.

(b) The events $B \cap C^c$, $B^c \cap C$, $B \cap C$, and $B^c \cap C^c$ form a partition of Ω, so by part (a), we have

$$\mathbf{P}(A) = \mathbf{P}(A \cap B \cap C^c) + \mathbf{P}(A \cap B^c \cap C) + \mathbf{P}(A \cap B \cap C) + \mathbf{P}(A \cap B^c \cap C^c). \quad (1)$$

The event $A \cap B$ can be written as the union of two disjoint events as follows:

$$A \cap B = (A \cap B \cap C) \cup (A \cap B \cap C^c),$$

so that

$$\mathbf{P}(A \cap B) = \mathbf{P}(A \cap B \cap C) + \mathbf{P}(A \cap B \cap C^c). \tag{2}$$

Similarly,

$$\mathbf{P}(A \cap C) = \mathbf{P}(A \cap B \cap C) + \mathbf{P}(A \cap B^c \cap C). \tag{3}$$

Combining Eqs. (1)-(3), we obtain the desired result.

Solution to Problem 1.10. Since the events $A \cap B^c$ and $A^c \cap B$ are disjoint, we have using the additivity axiom repeatedly,

$$\mathbf{P}\big((A\cap B^c)\cup(A^c\cap B)\big) = \mathbf{P}(A\cap B^c)+\mathbf{P}(A^c\cap B) = \mathbf{P}(A)-\mathbf{P}(A\cap B)+\mathbf{P}(B)-\mathbf{P}(A\cap B).$$

Solution to Problem 1.14. (a) Each possible outcome has probability $1/36$. There are 6 possible outcomes that are doubles, so the probability of doubles is $6/36 = 1/6$.

(b) The conditioning event (sum is 4 or less) consists of the 6 outcomes

$$\big\{(1,1),(1,2),(1,3),(2,1),(2,2),(3,1)\big\},$$

2 of which are doubles, so the conditional probability of doubles is $2/6 = 1/3$.

(c) There are 11 possible outcomes with at least one 6, namely, $(6,6)$, $(6,i)$, and $(i,6)$, for $i = 1, 2, \ldots, 5$. Thus, the probability that at least one die is a 6 is $11/36$.

(d) There are 30 possible outcomes where the dice land on different numbers. Out of these, there are 10 outcomes in which at least one of the rolls is a 6. Thus, the desired conditional probability is $10/30 = 1/3$.

Solution to Problem 1.15. Let A be the event that the first toss is a head and let B be the event that the second toss is a head. We must compare the conditional probabilities $\mathbf{P}(A \cap B \,|\, A)$ and $\mathbf{P}(A \cap B \,|\, A \cup B)$. We have

$$\mathbf{P}(A \cap B \,|\, A) = \frac{\mathbf{P}\big((A \cap B) \cap A\big)}{\mathbf{P}(A)} = \frac{\mathbf{P}(A \cap B)}{\mathbf{P}(A)},$$

and

$$\mathbf{P}(A \cap B \,|\, A \cup B) = \frac{\mathbf{P}\big((A \cap B) \cap (A \cup B)\big)}{\mathbf{P}(A \cup B)} = \frac{\mathbf{P}(A \cap B)}{\mathbf{P}(A \cup B)}.$$

Since $\mathbf{P}(A \cup B) \geq \mathbf{P}(A)$, the first conditional probability above is at least as large, so Alice is right, regardless of whether the coin is fair or not. In the case where the coin is fair, that is, if all four outcomes HH, HT, TH, TT are equally likely, we have

$$\frac{\mathbf{P}(A \cap B)}{\mathbf{P}(A)} = \frac{1/4}{1/2} = \frac{1}{2}, \qquad \frac{\mathbf{P}(A \cap B)}{\mathbf{P}(A \cup B)} = \frac{1/4}{3/4} = \frac{1}{3}.$$

A generalization of Alice's reasoning is that if A, B, and C are events such that $B \subset C$ and $A \cap B = A \cap C$ (for example, if $A \subset B \subset C$), then the event A is at least as likely if we know that B has occurred than if we know that C has occurred. Alice's reasoning corresponds to the special case where $C = A \cup B$.

Solution to Problem 1.16. In this problem, there is a tendency to reason that since the opposite face is either heads or tails, the desired probability is 1/2. This is, however, wrong, because given that heads came up, it is more likely that the two-headed coin was chosen. The correct reasoning is to calculate the conditional probability

$$\begin{aligned} p &= \mathbf{P}(\text{two-headed coin was chosen} \,|\, \text{heads came up}) \\ &= \frac{\mathbf{P}(\text{two-headed coin was chosen and heads came up})}{\mathbf{P}(\text{heads came up})}. \end{aligned}$$

We have

$$\mathbf{P}(\text{two-headed coin was chosen and heads came up}) = \frac{1}{3},$$

$$\mathbf{P}(\text{heads came up}) = \frac{1}{2},$$

so by taking the ratio of the above two probabilities, we obtain $p = 2/3$. Thus, the probability that the opposite face is tails is $1 - p = 1/3$.

Solution to Problem 1.17. Let A be the event that the batch will be accepted. Then $A = A_1 \cap A_2 \cap A_3 \cap A_4$, where A_i, $i = 1, \ldots, 4$, is the event that the ith item is not defective. Using the multiplication rule, we have

$$\mathbf{P}(A) = \mathbf{P}(A_1)\mathbf{P}(A_2 \,|\, A_1)\mathbf{P}(A_3 \,|\, A_1 \cap A_2)\mathbf{P}(A_4 \,|\, A_1 \cap A_2 \cap A_3) = \frac{95}{100} \cdot \frac{94}{99} \cdot \frac{93}{98} \cdot \frac{92}{97} = 0.812.$$

Solution to Problem 1.18. Using the definition of conditional probabilities, we have

$$\mathbf{P}(A \cap B \,|\, B) = \frac{\mathbf{P}(A \cap B \cap B)}{\mathbf{P}(B)} = \frac{\mathbf{P}(A \cap B)}{\mathbf{P}(B)} = \mathbf{P}(A \,|\, B).$$

Solution to Problem 1.19. Let A be the event that Alice does not find her paper in drawer i. Since the paper is in drawer i with probability p_i, and her search is successful with probability d_i, the multiplication rule yields $\mathbf{P}(A^c) = p_i d_i$, so that $\mathbf{P}(A) = 1 - p_i d_i$. Let B be the event that the paper is in drawer j. If $j \neq i$, then $A \cap B = B$, $\mathbf{P}(A \cap B) = \mathbf{P}(B)$, and we have

$$\mathbf{P}(B \,|\, A) = \frac{\mathbf{P}(A \cap B)}{\mathbf{P}(A)} = \frac{\mathbf{P}(B)}{\mathbf{P}(A)} = \frac{p_j}{1 - p_i d_i}.$$

Similarly, if $i = j$, we have

$$\mathbf{P}(B \,|\, A) = \frac{\mathbf{P}(A \cap B)}{\mathbf{P}(A)} = \frac{\mathbf{P}(B)\mathbf{P}(A \,|\, B)}{\mathbf{P}(A)} = \frac{p_i(1 - d_i)}{1 - p_i d_i}.$$

Solution to Problem 1.20. (a) Figure 1.1 provides a sequential description for the three different strategies. Here we assume 1 point for a win, 0 for a loss, and 1/2 point

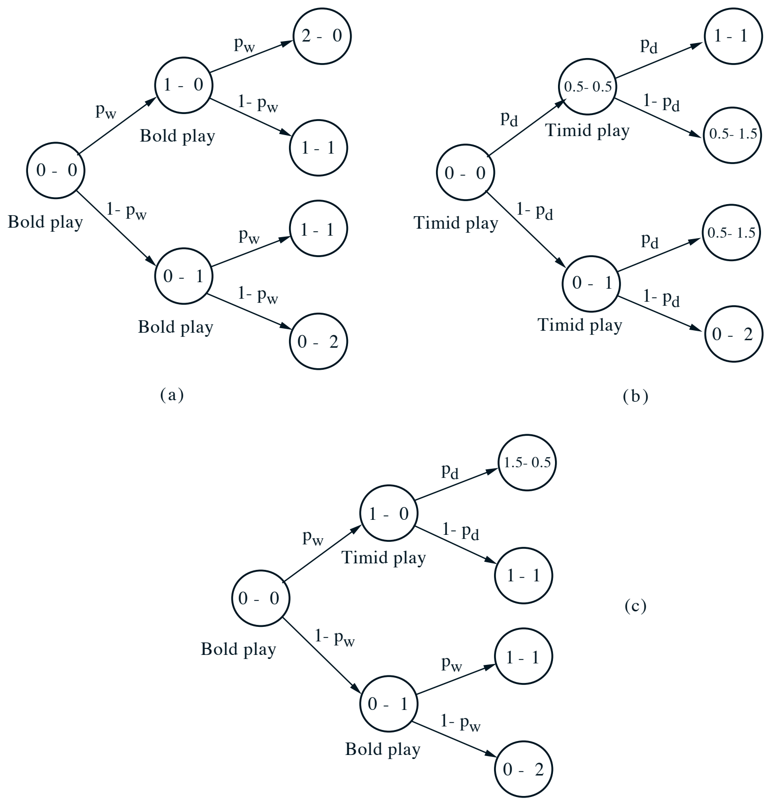

Figure 1.1: Sequential descriptions of the chess match histories under strategies (i), (ii), and (iii).

for a draw. In the case of a tied 1-1 score, we go to sudden death in the next game, and Boris wins the match (probability p_w), or loses the match (probability $1 - p_w$).

(i) Using the total probability theorem and the sequential description of Fig. 1.1(a), we have

$$\mathbf{P}(\text{Boris wins}) = p_w^2 + 2p_w(1 - p_w)p_w.$$

The term p_w^2 corresponds to the win-win outcome, and the term $2p_w(1-p_w)p_w$ corresponds to the win-lose-win and the lose-win-win outcomes.

(ii) Using Fig. 1.1(b), we have

$$\mathbf{P}(\text{Boris wins}) = p_d^2 p_w,$$

corresponding to the draw-draw-win outcome.

(iii) Using Fig. 1.1(c), we have

$$\mathbf{P}(\text{Boris wins}) = p_w p_d + p_w(1 - p_d)p_w + (1 - p_w)p_w^2.$$

The term $p_w p_d$ corresponds to the win-draw outcome, the term $p_w(1 - p_d)p_w$ corresponds to the win-lose-win outcome, and the term $(1 - p_w)p_w^2$ corresponds to lose-win-win outcome.

(b) If $p_w < 1/2$, Boris has a greater probability of losing rather than winning any one game, regardless of the type of play he uses. Despite this, the probability of winning the match with strategy (iii) can be greater than 1/2, provided that p_w is close enough to 1/2 and p_d is close enough to 1. As an example, if $p_w = 0.45$ and $p_d = 0.9$, with strategy (iii) we have

$$\mathbf{P}(\text{Boris wins}) = 0.45 \cdot 0.9 + 0.45^2 \cdot (1 - 0.9) + (1 - 0.45) \cdot 0.45^2 \approx 0.54.$$

With strategies (i) and (ii), the corresponding probabilities of a win can be calculated to be approximately 0.43 and 0.36, respectively. What is happening here is that with strategy (iii), Boris is allowed to select a playing style *after* seeing the result of the first game, while his opponent is not. Thus, by being able to dictate the playing style in each game after receiving partial information about the match's outcome, Boris gains an advantage.

Solution to Problem 1.21. Let $p(m, k)$ be the probability that the starting player wins when the jar initially contains m white and k black balls. We have, using the total probability theorem,

$$p(m, k) = \frac{m}{m+k} + \frac{k}{m+k}\big(1 - p(m, k-1)\big) = 1 - \frac{k}{m+k}p(m, k-1).$$

The probabilities $p(m, 1), p(m, 2), \ldots, p(m, n)$ can be calculated sequentially using this formula, starting with the initial condition $p(m, 0) = 1$.

Solution to Problem 1.22. We derive a recursion for the probability p_i that a white ball is chosen from the ith jar. We have, using the total probability theorem,

$$p_{i+1} = \frac{m+1}{m+n+1}p_i + \frac{m}{m+n+1}(1 - p_i) = \frac{1}{m+n+1}p_i + \frac{m}{m+n+1},$$

starting with the initial condition $p_1 = m/(m + n)$. Thus, we have

$$p_2 = \frac{1}{m+n+1} \cdot \frac{m}{m+n} + \frac{m}{m+n+1} = \frac{m}{m+n}.$$

More generally, this calculation shows that if $p_{i-1} = m/(m+n)$, then $p_i = m/(m+n)$. Thus, we obtain $p_i = m/(m + n)$ for all i.

Solution to Problem 1.23. Let $p_{i,n-i}(k)$ denote the probability that after k exchanges, a jar will contain i balls that started in that jar and $n - i$ balls that started in the other jar. We want to find $p_{n,0}(4)$. We argue recursively, using the total probability theorem. We have

$$\begin{aligned}
p_{n,0}(4) &= \frac{1}{n} \cdot \frac{1}{n} \cdot p_{n-1,1}(3), \\
p_{n-1,1}(3) &= p_{n,0}(2) + 2 \cdot \frac{n-1}{n} \cdot \frac{1}{n} \cdot p_{n-1,1}(2) + \frac{2}{n} \cdot \frac{2}{n} \cdot p_{n-2,2}(2), \\
p_{n,0}(2) &= \frac{1}{n} \cdot \frac{1}{n} \cdot p_{n-1,1}(1), \\
p_{n-1,1}(2) &= 2 \cdot \frac{n-1}{n} \cdot \frac{1}{n} \cdot p_{n-1,1}(1), \\
p_{n-2,2}(2) &= \frac{n-1}{n} \cdot \frac{n-1}{n} \cdot p_{n-1,1}(1), \\
p_{n-1,1}(1) &= 1.
\end{aligned}$$

Combining these equations, we obtain

$$p_{n,0}(4) = \frac{1}{n^2}\left(\frac{1}{n^2} + \frac{4(n-1)^2}{n^4} + \frac{4(n-1)^2}{n^4}\right) = \frac{1}{n^2}\left(\frac{1}{n^2} + \frac{8(n-1)^2}{n^4}\right).$$

Solution to Problem 1.24. Intuitively, there is something wrong with this rationale. The reason is that it is not based on a correctly specified probabilistic model. In particular, the event where both of the other prisoners are to be released is not properly accounted in the calculation of the posterior probability of release.

To be precise, let A, B, and C be the prisoners, and let A be the one who considers asking the guard. Suppose that all prisoners are a priori equally likely to be released. Suppose also that if B and C are to be released, then the guard chooses B or C with equal probability to reveal to A. Then, there are four possible outcomes:

(1) A and B are to be released, and the guard says B (probability 1/3).

(2) A and C are to be released, and the guard says C (probability 1/3).

(3) B and C are to be released, and the guard says B (probability 1/6).

(4) B and C are to be released, and the guard says C (probability 1/6).

Thus,

$$\begin{aligned}
\mathbf{P}(\text{A is to be released} \mid \text{guard says B}) &= \frac{\mathbf{P}(\text{A is to be released and guard says B})}{\mathbf{P}(\text{guard says B})} \\
&= \frac{1/3}{1/3 + 1/6} = \frac{2}{3}.
\end{aligned}$$

Similarly,

$$\mathbf{P}(\text{A is to be released} \mid \text{guard says C}) = \frac{2}{3}.$$

Thus, regardless of the identity revealed by the guard, the probability that A is released is equal to 2/3, the a priori probability of being released.

Solution to Problem 1.25. Let $\overline{m}$ and $\underline{m}$ be the larger and the smaller of the two amounts, respectively. Consider the three events

$$A = \{X < \underline{m}), \qquad B = \{\underline{m} < X < \overline{m}), \qquad C = \{\overline{m} < X).$$

Let $\overline{A}$ (or $\overline{B}$ or $\overline{C}$) be the event that A (or B or C, respectively) occurs *and* you first select the envelope containing the larger amount $\overline{m}$. Let $\underline{A}$ (or $\underline{B}$ or $\underline{C}$) be the event that A (or B or C, respectively) occurs *and* you first select the envelope containing the smaller amount $\underline{m}$. Finally, consider the event

$$W = \{\text{you end up with the envelope containing } \overline{m}\}.$$

We want to determine $\mathbf{P}(W)$ and check whether it is larger than 1/2 or not.

By the total probability theorem, we have

$$\mathbf{P}(W \mid A) = \frac{1}{2}\big(\mathbf{P}(W \mid \overline{A}) + \mathbf{P}(W \mid \underline{A})\big) = \frac{1}{2}(1+0) = \frac{1}{2},$$

$$\mathbf{P}(W \mid B) = \frac{1}{2}\big(\mathbf{P}(W \mid \overline{B}) + \mathbf{P}(W \mid \underline{B})\big) = \frac{1}{2}(1+1) = 1,$$

$$\mathbf{P}(W \mid C) = \frac{1}{2}\big(\mathbf{P}(W \mid \overline{C}) + \mathbf{P}(W \mid \underline{C})\big) = \frac{1}{2}(0+1) = \frac{1}{2}.$$

Using these relations together with the total probability theorem, we obtain

$$\begin{aligned}\mathbf{P}(W) &= \mathbf{P}(A)\mathbf{P}(W \mid A) + \mathbf{P}(B)\mathbf{P}(W \mid B) + \mathbf{P}(C)\mathbf{P}(W \mid C) \\ &= \frac{1}{2}\big(\mathbf{P}(A) + \mathbf{P}(B) + \mathbf{P}(C)\big) + \frac{1}{2}\mathbf{P}(B) \\ &= \frac{1}{2} + \frac{1}{2}\mathbf{P}(B).\end{aligned}$$

Since $\mathbf{P}(B) > 0$ by assumption, it follows that $\mathbf{P}(W) > 1/2$, so your friend is correct.

Solution to Problem 1.26. (a) We use the formula

$$\mathbf{P}(A \mid B) = \frac{\mathbf{P}(A \cap B)}{\mathbf{P}(B)} = \frac{\mathbf{P}(A)\mathbf{P}(B \mid A)}{\mathbf{P}(B)}.$$

Since all crows are black, we have $\mathbf{P}(B) = 1 - q$. Furthermore, $\mathbf{P}(A) = p$. Finally, $\mathbf{P}(B \mid A) = 1 - q = \mathbf{P}(B)$, since the probability of observing a (black) crow is not affected by the truth of our hypothesis. We conclude that $\mathbf{P}(A \mid B) = \mathbf{P}(A) = p$. Thus, the new evidence, while compatible with the hypothesis "all cows are white," does not change our beliefs about its truth.

(b) Once more,

$$\mathbf{P}(A \mid C) = \frac{\mathbf{P}(A \cap C)}{\mathbf{P}(C)} = \frac{\mathbf{P}(A)\mathbf{P}(C \mid A)}{\mathbf{P}(C)}.$$

Given the event A, a cow is observed with probability q, and it must be white. Thus, $\mathbf{P}(C \mid A) = q$. Given the event A^c, a cow is observed with probability q, and it is white with probability 1/2. Thus, $\mathbf{P}(C \mid A^c) = q/2$. Using the total probability theorem,

$$\mathbf{P}(C) = \mathbf{P}(A)\mathbf{P}(C \mid A) + \mathbf{P}(A^c)\mathbf{P}(C \mid A^c) = pq + (1-p)\frac{q}{2}.$$

Hence,

$$\mathbf{P}(A \mid C) = \frac{pq}{pq + (1-p)\frac{q}{2}} = \frac{2p}{1+p} > p.$$

Thus, the observation of a white cow makes the hypothesis "all cows are white" more likely to be true.

Solution to Problem 1.27. Since Bob tosses one more coin that Alice, it is impossible that they toss both the same number of heads and the same number of tails. So Bob tosses either more heads than Alice or more tails than Alice (but not both). Since the coins are fair, these events are equally likely by symmetry, so both events have probability 1/2.

An alternative solution is to argue that if Alice and Bob are tied after $2n$ tosses, they are equally likely to win. If they are not tied, then their scores differ by at least 2, and toss $2n+1$ will not change the final outcome. This argument may also be expressed algebraically by using the total probability theorem. Let B be the event that Bob tosses more heads. Let X be the event that after each has tossed n of their coins, Bob has more heads than Alice, let Y be the event that under the same conditions, Alice has more heads than Bob, and let Z be the event that they have the same number of heads. Since the coins are fair, we have $\mathbf{P}(X) = \mathbf{P}(Y)$, and also $\mathbf{P}(Z) = 1 - \mathbf{P}(X) - \mathbf{P}(Y)$. Furthermore, we see that

$$\mathbf{P}(B \mid X) = 1, \qquad \mathbf{P}(B \mid Y) = 0, \qquad \mathbf{P}(B \mid Z) = \frac{1}{2}.$$

Now we have, using the total probability theorem,

$$\begin{aligned}\mathbf{P}(B) &= \mathbf{P}(X) \cdot \mathbf{P}(B \mid X) + \mathbf{P}(Y) \cdot \mathbf{P}(B \mid Y) + \mathbf{P}(Z) \cdot \mathbf{P}(B \mid Z) \\ &= \mathbf{P}(X) + \frac{1}{2} \cdot \mathbf{P}(Z) \\ &= \frac{1}{2} \cdot \big(\mathbf{P}(X) + \mathbf{P}(Y) + \mathbf{P}(Z)\big) \\ &= \frac{1}{2}.\end{aligned}$$

as required.

Solution to Problem 1.30. Consider the sample space for the hunter's strategy. The events that lead to the correct path are:

(1) Both dogs agree on the correct path (probability p^2, by independence).

(2) The dogs disagree, dog 1 chooses the correct path, and hunter follows dog 1 [probability $p(1-p)/2$].

(3) The dogs disagree, dog 2 chooses the correct path, and hunter follows dog 2 [probability $p(1-p)/2$].

The above events are disjoint, so we can add the probabilities to find that under the hunter's strategy, the probability that he chooses the correct path is

$$p^2 + \frac{1}{2}p(1-p) + \frac{1}{2}p(1-p) = p.$$

On the other hand, if the hunter lets one dog choose the path, this dog will also choose the correct path with probability p. Thus, the two strategies are equally effective.

Solution to Problem 1.31. (a) Let A be the event that a 0 is transmitted. Using the total probability theorem, the desired probability is

$$\mathbf{P}(A)(1-\epsilon_0) + \big(1-\mathbf{P}(A)\big)(1-\epsilon_1) = p(1-\epsilon_0) + (1-p)(1-\epsilon_1).$$

(b) By independence, the probability that the string 1011 is received correctly is

$$(1-\epsilon_0)(1-\epsilon_1)^3.$$

(c) In order for a 0 to be decoded correctly, the received string must be 000, 001, 010, or 100. Given that the string transmitted was 000, the probability of receiving 000 is $(1-\epsilon_0)^3$, and the probability of each of the strings 001, 010, and 100 is $\epsilon_0(1-\epsilon_0)^2$. Thus, the probability of correct decoding is

$$3\epsilon_0(1-\epsilon_0)^2 + (1-\epsilon_0)^3.$$

(d) When the symbol is 0, the probabilities of correct decoding with and without the scheme of part (c) are $3\epsilon_0(1-\epsilon_0)^2 + (1-\epsilon_0)^3$ and $1-\epsilon_0$, respectively. Thus, the probability is improved with the scheme of part (c) if

$$3\epsilon_0(1-\epsilon_0)^2 + (1-\epsilon_0)^3 > (1-\epsilon_0),$$

or

$$(1-\epsilon_0)(1+2\epsilon_0) > 1,$$

which is equivalent to $\epsilon_0 < 1/2$.

(e) Using Bayes' rule, we have

$$\mathbf{P}(0\,|\,101) = \frac{\mathbf{P}(0)\mathbf{P}(101\,|\,0)}{\mathbf{P}(0)\mathbf{P}(101\,|\,0) + \mathbf{P}(1)\mathbf{P}(101\,|\,1)}.$$

The probabilities needed in the above formula are

$$\mathbf{P}(0) = p, \quad \mathbf{P}(1) = 1-p, \quad \mathbf{P}(101\,|\,0) = \epsilon_0^2(1-\epsilon_0), \quad \mathbf{P}(101\,|\,1) = \epsilon_1(1-\epsilon_1)^2.$$

Solution to Problem 1.32. The answer to this problem is not unique and depends on the assumptions we make on the reproductive strategy of the king's parents.

Suppose that the king's parents had decided to have exactly two children and then stopped. There are four possible and equally likely outcomes, namely BB, GG, BG, and GB (B stands for "boy" and G stands for "girl"). Given that at least one child was a boy (the king), the outcome GG is eliminated and we are left with three equally likely outcomes (BB, BG, and GB). The probability that the sibling is male (the conditional probability of BB) is 1/3 .

Suppose on the other hand that the king's parents had decided to have children until they would have a male child. In that case, the king is the second child, and the sibling is female, with certainty.

Solution to Problem 1.33. Flip the coin twice. If the outcome is heads-tails, choose the opera. if the outcome is tails-heads, choose the movies. Otherwise, repeat the process, until a decision can be made. Let A_k be the event that a decision was made at the kth round. Conditional on the event A_k, the two choices are equally likely, and we have

$$\mathbf{P}(\text{opera}) = \sum_{k=1}^{\infty} \mathbf{P}(\text{opera} \,|\, A_k)\mathbf{P}(A_k) = \sum_{k=1}^{\infty} \frac{1}{2}\mathbf{P}(A_k) = \frac{1}{2}.$$

We have used here the property $\sum_{k=0}^{\infty} \mathbf{P}(A_k) = 1$, which is true as long as $\mathbf{P}(\text{heads}) > 0$ and $\mathbf{P}(\text{tails}) > 0$.

Solution to Problem 1.34. The system may be viewed as a series connection of three subsystems, denoted 1, 2, and 3 in Fig. 1.19 in the text. The probability that the entire system is operational is $p_1p_2p_3$, where p_i is the probability that subsystem i is operational. Using the formulas for the probability of success of a series or a parallel system given in Example 1.24, we have

$$p_1 = p, \qquad p_3 = 1 - (1-p)^2,$$

and

$$p_2 = 1 - (1-p)\big(1 - p\big(1 - (1-p)^3\big)\big).$$

Solution to Problem 1.35. Let A_i be the event that exactly i components are operational. The probability that the system is operational is the probability of the union $\cup_{i=k}^{n} A_i$, and since the A_i are disjoint, it is equal to

$$\sum_{i=k}^{n} \mathbf{P}(A_i) = \sum_{i=k}^{n} p(i),$$

where $p(i)$ are the binomial probabilities. Thus, the probability of an operational system is

$$\sum_{i=k}^{n} \binom{n}{i} p^i (1-p)^{n-i}.$$

Solution to Problem 1.36. (a) Let A denote the event that the city experiences a black-out. Since the power plants fail independent of each other, we have

$$\mathbf{P}(A) = \prod_{i=1}^{n} p_i.$$

(b) There will be a black-out if either all n or any $n-1$ power plants fail. These two events are disjoint, so we can calculate the probability $\mathbf{P}(A)$ of a black-out by adding their probabilities:

$$\mathbf{P}(A) = \prod_{i=1}^{n} p_i + \sum_{i=1}^{n} \left((1-p_i) \prod_{j \neq i} p_j \right).$$

Here, $(1-p_i)\prod_{j\neq i} p_j$ is the probability that $n-1$ plants have failed and plant i is the one that has not failed.

Solution to Problem 1.37. The probability that k_1 voice users and k_2 data users simultaneously need to be connected is $p_1(k_1)p_2(k_2)$, where $p_1(k_1)$ and $p_2(k_2)$ are the corresponding binomial probabilities, given by

$$p_i(k_i) = \binom{n_i}{k_i} p_i^{k_i}(1-p_i)^{n_i-k_i}, \qquad i = 1, 2.$$

The probability that more users want to use the system than the system can accommodate is the sum of all products $p_1(k_1)p_2(k_2)$ as k_1 and k_2 range over all possible values whose total bit rate requirement $k_1r_1+k_2r_2$ exceeds the capacity c of the system. Thus, the desired probability is

$$\sum_{\{(k_1,k_2)\,|\,k_1r_1+k_2r_2>c,\,k_1\leq n_1,\,k_2\leq n_2\}} p_1(k_1)p_2(k_2).$$

Solution to Problem 1.38. We have

$$p_T = \mathbf{P}(\text{at least 6 out of the 8 remaining holes are won by Telis}),$$

$$p_W = \mathbf{P}(\text{at least 4 out of the 8 remaining holes are won by Wendy}).$$

Using the binomial formulas,

$$p_T = \sum_{k=6}^{8} \binom{8}{k} p^k(1-p)^{8-k}, \qquad p_W = \sum_{k=4}^{8} \binom{8}{k} (1-p)^k p^{8-k}.$$

The amount of money that Telis should get is $10 \cdot p_T/(p_T + p_W)$ dollars.

Solution to Problem 1.39. Let the event A be the event that the professor teaches her class, and let B be the event that the weather is bad. We have

$$\mathbf{P}(A) = \mathbf{P}(B)\mathbf{P}(A\,|\,B) + \mathbf{P}(B^c)\mathbf{P}(A\,|\,B^c),$$

and

$$\mathbf{P}(A\,|\,B) = \sum_{i=k}^{n} \binom{n}{i} p_b^i(1-p_b)^{n-i},$$

$$\mathbf{P}(A\,|\,B^c) = \sum_{i=k}^{n} \binom{n}{i} p_g^i(1-p_g)^{n-i}.$$

Therefore,

$$\mathbf{P}(A) = \mathbf{P}(B)\sum_{i=k}^{n} \binom{n}{i} p_b^i(1-p_b)^{n-i} + \big(1-\mathbf{P}(B)\big)\sum_{i=k}^{n} \binom{n}{i} p_g^i(1-p_g)^{n-i}.$$

Solution to Problem 1.40. Let A be the event that the first $n-1$ tosses produce an even number of heads, and let E be the event that the nth toss is a head. We can obtain an even number of heads in n tosses in two distinct ways: 1) there is an even number of heads in the first $n-1$ tosses, and the nth toss results in tails: this is the event $A \cap E^c$; 2) there is an odd number of heads in the first $n-1$ tosses, and the nth toss results in heads: this is the event $A^c \cap E$. Using also the independence of A and E,

$$\begin{aligned} q_n &= \mathbf{P}\big((A \cap E^c) \cup (A^c \cap E)\big) \\ &= \mathbf{P}(A \cap E^c) + \mathbf{P}(A^c \cap E) \\ &= \mathbf{P}(A)\mathbf{P}(E^c) + \mathbf{P}(A^c)\mathbf{P}(E) \\ &= (1-p)q_{n-1} + p(1-q_{n-1}). \end{aligned}$$

We now use induction. For $n = 0$, we have $q_0 = 1$, which agrees with the given formula for q_n. Assume, that the formula holds with n replaced by $n-1$, i.e.,

$$q_{n-1} = \frac{1 + (1-2p)^{n-1}}{2}.$$

Using this equation, we have

$$\begin{aligned} q_n &= p(1-q_{n-1}) + (1-p)q_{n-1} \\ &= p + (1-2p)q_{n-1} \\ &= p + (1-2p)\frac{1 + (1-2p)^{n-1}}{2} \\ &= \frac{1 + (1-2p)^n}{2}, \end{aligned}$$

so the given formula holds for all n.

Solution to Problem 1.41. We have

$$\mathbf{P}(N = n) = \mathbf{P}(A_{1,n-1} \cap A_{n,n}) = \mathbf{P}(A_{1,n-1})\mathbf{P}(A_{n,n} \mid A_{1,n-1}),$$

where for $i \leq j$, $A_{i,j}$ is the event that contestant i's number is the smallest of the numbers of contestants $1, \ldots, j$. We also have

$$\mathbf{P}(A_{1,n-1}) = \frac{1}{n-1}.$$

We claim that

$$\mathbf{P}(A_{n,n} \mid A_{1,n-1}) = \mathbf{P}(A_{n,n}) = \frac{1}{n}.$$

The reason is that by symmetry, we have

$$\mathbf{P}(A_{n,n} \mid A_{i,n-1}) = \mathbf{P}(A_{n,n} \mid A_{1,n-1}), \qquad i = 1, \ldots, n-1,$$

while by the total probability theorem,

$$\begin{aligned} \mathbf{P}(A_{n,n}) &= \sum_{i=1}^{n-1} \mathbf{P}(A_{i,n-1})\mathbf{P}(A_{n,n} \mid A_{i,n-1}) \\ &= \mathbf{P}(A_{n,n} \mid A_{1,n-1}) \sum_{i=1}^{n-1} \mathbf{P}(A_{i,n-1}) \\ &= \mathbf{P}(A_{n,n} \mid A_{1,n-1}). \end{aligned}$$

Hence

$$\mathbf{P}(N = n) = \frac{1}{n-1} \cdot \frac{1}{n}.$$

An alternative solution is also possible, using the counting methods developed in Section 1.6. Let us fix a particular choice of n. Think of an outcome of the experiment as an ordering of the values of the n contestants, so that there are $n!$ equally likely outcomes. The event $\{N = n\}$ occurs if and only if the first contestant's number is smallest among the first $n-1$ contestants, and contestant n's number is the smallest among the first n contestants. This event can occur in $(n-2)!$ different ways, namely, all the possible ways of ordering contestants $2, \ldots, n-1$. Thus, the probability of this event is $(n-2)!/n! = 1/(n(n-1))$, in agreement with the previous solution.

Solution to Problem 1.49. A sum of 11 is obtained with the following 6 combinations:

$$(6,4,1)\ (6,3,2)\ (5,5,1)\ (5,4,2)\ (5,3,3)\ (4,4,3).$$

A sum of 12 is obtained with the following 6 combinations:

$$(6,5,1)\ (6,4,2)\ (6,3,3)\ (5,5,2)\ (5,4,3)\ (4,4,4).$$

Each combination of 3 distinct numbers corresponds to 6 permutations, while each combination of 3 numbers, two of which are equal, corresponds to 3 permutations. Counting the number of permutations in the 6 combinations corresponding to a sum of 11, we obtain $6+6+3+6+3+3 = 27$ permutations. Counting the number of permutations in the 6 combinations corresponding to a sum of 12, we obtain $6+6+3+3+6+1 = 25$ permutations. Since all permutations are equally likely, a sum of 11 is more likely than a sum of 12.

Note also that the sample space has $6^3 = 216$ elements, so we have $\mathbf{P}(11) = 27/216$, $\mathbf{P}(12) = 25/216$.

Solution to Problem 1.50. The sample space consists of all possible choices for the birthday of each person. Since there are n persons, and each has 365 choices for their birthday, the sample space has 365^n elements. Let us now consider those choices of birthdays for which no two persons have the same birthday. Assuming that $n \leq 365$, there are 365 choices for the first person, 364 for the second, etc., for a total of $365 \cdot 364 \cdots (365 - n + 1)$. Thus,

$$\mathbf{P}(\text{no two birthdays coincide}) = \frac{365 \cdot 364 \cdots (365 - n + 1)}{365^n}.$$

It is interesting to note that for n as small as 23, the probability that there are two persons with the same birthday is larger than 1/2.

Solution to Problem 1.51. (a) We number the red balls from 1 to m, and the white balls from $m+1$ to $m+n$. One possible sample space consists of all pairs of integers (i, j) with $1 \leq i, j \leq m+n$ and $i \neq j$. The total number of possible outcomes is $(m+n)(m+n-1)$. The number of outcomes corresponding to red-white selection, (i.e., $i \in \{1, \ldots, m\}$ and $j \in \{m+1, \ldots, m+n\}$) is mn. The number of outcomes corresponding to white-red selection, (i.e., $i \in \{m+1, \ldots, m+n\}$ and $j \in \{1, \ldots, m\}$) is also mn. Thus, the desired probability that the balls are of different color is

$$\frac{2mn}{(m+n)(m+n-1)}.$$

Another possible sample space consists of all the possible ordered color pairs, i.e., $\{RR, RW, WR, WW\}$. We then have to calculate the probability of the event $\{RW, WR\}$. We consider a sequential description of the experiment, i.e., we first select the first ball and then the second. In the first stage, the probability of a red ball is $m/(m+n)$. In the second stage, the probability of a red ball is either $m/(m+n-1)$ or $(m-1)/(m+n-1)$ depending on whether the first ball was white or red, respectively. Therefore, using the multiplication rule, we have

$$\mathbf{P}(RR) = \frac{m}{m+n} \cdot \frac{m-1}{m-1+n}, \qquad \mathbf{P}(RW) = \frac{m}{m+n} \cdot \frac{n}{m-1+n},$$

$$\mathbf{P}(WR) = \frac{n}{m+n} \cdot \frac{m}{m+n-1}, \qquad \mathbf{P}(WW) = \frac{n}{m+n} \cdot \frac{n-1}{m+n-1}.$$

The desired probability is

$$\begin{aligned}\mathbf{P}\big(\{RW, WR\}\big) &= \mathbf{P}(RW) + \mathbf{P}(WR) \\ &= \frac{m}{m+n} \cdot \frac{n}{m-1+n} + \frac{n}{m+n} \cdot \frac{m}{m+n-1} \\ &= \frac{2mn}{(m+n)(m+n-1)}.\end{aligned}$$

(b) We calculate the conditional probability of all balls being red, given any of the possible values of k. We have $\mathbf{P}(R \,|\, k = 1) = m/(m + n)$ and, as found in part (a), $\mathbf{P}(RR \,|\, k = 2) = m(m - 1)/(m + n)(m - 1 + n)$. Arguing sequentially as in part (a), we also have $\mathbf{P}(RRR \,|\, k = 3) = m(m - 1)(m - 2)/(m + n)(m - 1 + n)(m - 2 + n)$. According to the total probability theorem, the desired answer is

$$\frac{1}{3}\left(\frac{m}{m+n} + \frac{m(m-1)}{(m+n)(m-1+n)} + \frac{m(m-1)(m-2)}{(m+n)(m-1+n)(m-2+n)}\right).$$

Solution to Problem 1.52. The probability that the 13th card is the first king to be dealt is the probability that out of the first 13 cards to be dealt, exactly one was a king, and that the king was dealt last. Now, given that exactly one king was dealt in the first 13 cards, the probability that the king was dealt last is just 1/13, since each "position" is equally likely. Thus, it remains to calculate the probability that there was exactly one king in the first 13 cards dealt. To calculate this probability we count the "favorable" outcomes and divide by the total number of possible outcomes. We first count the favorable outcomes, namely those with exactly one king in the first 13 cards dealt. We can choose a particular king in 4 ways, and we can choose the other 12 cards in $\binom{48}{12}$ ways, therefore there are $4 \cdot \binom{48}{12}$ favorable outcomes. There are $\binom{52}{13}$ total outcomes, so the desired probability is

$$\frac{1}{13} \cdot \frac{4 \cdot \dbinom{48}{12}}{\dbinom{52}{13}}.$$

For an alternative solution, we argue as in Example 1.10. The probability that the first card is not a king is 48/52. Given that, the probability that the second is not a king is 47/51. We continue similarly until the 12th card. The probability that the 12th card is not a king, given that none of the preceding 11 was a king, is 37/41. (There are $52-11=41$ cards left, and $48-11=37$ of them are not kings.) Finally, the conditional probability that the 13th card is a king is 4/40. The desired probability is

$$\frac{48 \cdot 47 \cdots 37 \cdot 4}{52 \cdot 51 \cdots 41 \cdot 40}.$$

Solution to Problem 1.53. Suppose we label the classes A, B, and C. The probability that Joe and Jane will both be in class A is the number of possible combinations for class A that involve both Joe and Jane, divided by the total number of combinations for class A. Therefore, this probability is

$$\frac{\binom{88}{28}}{\binom{90}{30}}.$$

Since there are three classes, the probability that Joe and Jane end up in the same class is

$$3 \cdot \frac{\binom{88}{28}}{\binom{90}{30}}.$$

A much simpler solution is as follows. We place Joe in one class. Regarding Jane, there are 89 possible "slots", and only 29 of them place her in the same class as Joe. Thus, the answer is 29/89, which turns out to agree with the answer obtained earlier.

Solution to Problem 1.54. (a) Since the cars are all distinct, there are 20! ways to line them up.

(b) To find the probability that the cars will be parked so that they alternate, we count the number of "favorable" outcomes, and divide by the total number of possible outcomes found in part (a). We count in the following manner. We first arrange the US cars in an ordered sequence (permutation). We can do this in 10! ways, since there are 10 distinct cars. Similarly, arrange the foreign cars in an ordered sequence, which can also be done in 10! ways. Finally, interleave the two sequences. This can be done in two different ways, since we can let the first car be either US-made or foreign. Thus, we have a total of $2 \cdot 10! \cdot 10!$ possibilities, and the desired probability is

$$\frac{2 \cdot 10! \cdot 10!}{20!}.$$

Note that we could have solved the second part of the problem by neglecting the fact that the cars are distinct. Suppose the foreign cars are indistinguishable, and also that the US cars are indistinguishable. Out of the 20 available spaces, we need to choose 10 spaces in which to place the US cars, and thus there are $\binom{20}{10}$ possible outcomes. Out of these outcomes, there are only two in which the cars alternate, depending on whether we start with a US or a foreign car. Thus, the desired probability is $2/\binom{20}{10}$, which coincides with our earlier answer.

Solution to Problem 1.55. We count the number of ways in which we can safely place 8 distinguishable rooks, and then divide this by the total number of possibilities. First we count the number of favorable positions for the rooks. We will place the rooks one by one on the 8×8 chessboard. For the first rook, there are no constraints, so we have 64 choices. Placing this rook, however, eliminates one row and one column. Thus, for the second rook, we can imagine that the illegal column and row have been removed, thus leaving us with a 7×7 chessboard, and with 49 choices. Similarly, for the third rook we have 36 choices, for the fourth 25, etc. In the absence of any restrictions, there are $64 \cdot 63 \cdots 57 = 64!/56!$ ways we can place 8 rooks, so the desired probability is

$$\frac{64 \cdot 49 \cdot 36 \cdot 25 \cdot 16 \cdot 9 \cdot 4}{\frac{64!}{56!}}.$$

Solution to Problem 1.56. (a) There are $\binom{8}{4}$ ways to pick 4 lower level classes, and $\binom{10}{3}$ ways to choose 3 higher level classes, so there are

$$\binom{8}{4}\binom{10}{3}$$

valid curricula.

(b) This part is more involved. We need to consider several different cases:

(i) Suppose we do not choose L_1. Then both L_2 and L_3 must be chosen; otherwise no higher level courses would be allowed. Thus, we need to choose 2 more lower level classes out of the remaining 5, and 3 higher level classes from the available 5. We then obtain $\binom{5}{2}\binom{5}{3}$ valid curricula.

(ii) If we choose L_1 but choose neither L_2 nor L_3, we have $\binom{5}{3}\binom{5}{3}$ choices.

(iii) If we choose L_1 and choose one of L_2 or L_3, we have $2 \cdot \binom{5}{2}\binom{5}{3}$ choices. This is because there are two ways of choosing between L_2 and L_3, $\binom{5}{2}$ ways of choosing 2 lower level classes from $L_4, \ldots, L_8$, and $\binom{5}{3}$ ways of choosing 3 higher level classes from $H_1, \ldots, H_5$.

(iv) Finally, if we choose L_1, L_2, and L_3, we have $\binom{5}{1}\binom{10}{3}$ choices.

Note that we are not double counting, because there is no overlap in the cases we are considering, and furthermore we have considered every possible choice. The total is obtained by adding the counts for the above four cases.

Solution to Problem 1.57. Let us fix the order in which letters appear in the sentence. There are 26! choices, corresponding to the possible permutations of the 26-letter alphabet. Having fixed the order of the letters, we need to separate them into words. To obtain 6 words, we need to place 5 separators ("blanks") between the letters. With 26 letters, there are 25 possible positions for these blanks, and the number of choices is $\binom{25}{5}$. Thus, the desired number of sentences is $25!\binom{25}{5}$. Generalizing, the number of sentences consisting of w nonempty words using exactly once each letter from a l-letter alphabet is equal to

$$l!\binom{l-1}{w-1}.$$

Solution to Problem 1.58. (a) The sample space consists of all ways of drawing 7 elements out of a 52-element set, so it contains $\binom{52}{7}$ possible outcomes. Let us count those outcomes that involve exactly 3 aces. We are free to select any 3 out of the 4 aces, and any 4 out of the 48 remaining cards, for a total of $\binom{4}{3}\binom{48}{4}$ choices. Thus,

$$\mathbf{P}(\text{7 cards include exactly 3 aces}) = \frac{\binom{4}{3}\binom{48}{4}}{\binom{52}{7}}.$$

(b) Proceeding similar to part (a), we obtain

$$\mathbf{P}(\text{7 cards include exactly 2 kings}) = \frac{\binom{4}{2}\binom{48}{5}}{\binom{52}{7}}.$$

(c) If A and B stand for the events in parts (a) and (b), respectively, we are looking for $\mathbf{P}(A \cup B) = \mathbf{P}(A) + \mathbf{P}(B) - \mathbf{P}(A \cap B)$. The event $A \cap B$ (having exactly 3 aces and exactly 2 kings) can occur by choosing 3 out of the 4 available aces, 2 out of the 4 available kings, and 2 more cards out of the remaining 44. Thus, this event consists of $\binom{4}{3}\binom{4}{2}\binom{44}{2}$ distinct outcomes. Hence,

$$\mathbf{P}(\text{7 cards include 3 aces and/or 2 kings}) = \frac{\binom{4}{3}\binom{48}{4} + \binom{4}{2}\binom{48}{5} - \binom{4}{3}\binom{4}{2}\binom{44}{2}}{\binom{52}{7}}.$$

Solution to Problem 1.59. Clearly if $n > m$, or $n > k$, or $m - n > 100 - k$, the probability must be zero. If $n \le m$, $n \le k$, and $m - n \le 100 - k$, then we can find the probability that the test drive found n of the 100 cars defective by counting the total number of size m subsets, and then the number of size m subsets that contain n lemons. Clearly, there are $\binom{100}{m}$ different subsets of size m. To count the number of size m subsets with n lemons, we first choose n lemons from the k available lemons, and then choose $m - n$ good cars from the $100 - k$ available good cars. Thus, the number of ways to choose a subset of size m from 100 cars, and get n lemons, is

$$\binom{k}{n}\binom{100-k}{m-n},$$

and the desired probability is

$$\frac{\binom{k}{n}\binom{100-k}{m-n}}{\binom{100}{m}}.$$

Solution to Problem 1.60. The size of the sample space is the number of different ways that 52 objects can be divided in 4 groups of 13, and is given by the multinomial formula

$$\frac{52!}{13!\,13!\,13!\,13!}.$$

There are 4! different ways of distributing the 4 aces to the 4 players, and there are

$$\frac{48!}{12!\,12!\,12!\,12!}$$

different ways of dividing the remaining 48 cards into 4 groups of 12. Thus, the desired probability is

$$\frac{4!\,\dfrac{48!}{12!\,12!\,12!\,12!}}{\dfrac{52!}{13!\,13!\,13!\,13!}}.$$

An alternative solution can be obtained by considering a different, but probabilistically equivalent method of dealing the cards. Each player has 13 slots, each one of which is to receive one card. Instead of shuffling the deck, we place the 4 aces at the top, and start dealing the cards one at a time, with each free slot being equally likely to receive the next card. For the event of interest to occur, the first ace can go anywhere; the second can go to any one of the 39 slots (out of the 51 available) that correspond to players that do not yet have an ace; the third can go to any one of the 26 slots (out of the 50 available) that correspond to the two players that do not yet have an ace; and finally, the fourth, can go to any one of the 13 slots (out of the 49 available) that correspond to the only player who does not yet have an ace. Thus, the desired probability is

$$\frac{39 \cdot 26 \cdot 13}{51 \cdot 50 \cdot 49}.$$

By simplifying our previous answer, it can be checked that it is the same as the one obtained here, thus corroborating the intuitive fact that the two different ways of dealing the cards are probabilistically equivalent.