CHAPTER 3

Solution to Problem 3.1. The random variable $Y = g(X)$ is discrete and its PMF is given by

$$p_Y(1) = \mathbf{P}(X \leq 1/3) = 1/3, \qquad p_Y(2) = 1 - p_Y(1) = 2/3.$$

Thus,

$$\mathbf{E}[Y] = \frac{1}{3} \cdot 1 + \frac{2}{3} \cdot 2 = \frac{5}{3}.$$

The same result is obtained using the expected value rule:

$$\mathbf{E}[Y] = \int_0^1 g(x) f_X(x)\, dx = \int_0^{1/3} dx + \int_{1/3}^1 2\, dx = \frac{5}{3}.$$

Solution to Problem 3.2. We have

$$\int_{-\infty}^{\infty} f_X(x) dx = \int_{-\infty}^{\infty} \frac{\lambda}{2} e^{-\lambda |x|}\, dx = 2 \cdot \frac{1}{2} \int_0^{\infty} \lambda e^{-\lambda x}\, dx = 2 \cdot \frac{1}{2} = 1,$$

where we have used the fact $\int_0^{\infty} \lambda e^{-\lambda x} dx = 1$, i.e., the normalization property of the exponential PDF. By symmetry of the PDF, we have $\mathbf{E}[X] = 0$. We also have

$$\mathbf{E}[X^2] = \int_{-\infty}^{\infty} x^2 \frac{\lambda}{2} e^{-\lambda |x|} dx = \int_0^{\infty} x^2 \lambda e^{-\lambda x} dx = \frac{2}{\lambda^2},$$

where we have used the fact that the second moment of the exponential PDF is $2/\lambda^2$. Thus

$$\text{var}(X) = \mathbf{E}[X^2] - \big(\mathbf{E}[X]\big)^2 = 2/\lambda^2.$$

Solution to Problem 3.5. Let $A = bh/2$ be the area of the given triangle, where b is the length of the base, and h is the height of the triangle. From the randomly chosen point, draw a line parallel to the base, and let A_x be the area of the triangle thus formed. The height of this triangle is $h - x$ and its base has length $b(h - x)/h$. Thus $A_x = b(h - x)^2/(2h)$. For $x \in [0, h]$, we have

$$F_X(x) = 1 - \mathbf{P}(X > x) = 1 - \frac{A_x}{A} = 1 - \frac{b(h-x)^2/(2h)}{bh/2} = 1 - \left(\frac{h-x}{h}\right)^2,$$

while $F_X(x) = 0$ for $x < 0$ and $F_X(x) = 1$ for $x > h$.

The PDF is obtained by differentiating the CDF. We have

$$f_X(x) = \frac{dF_X}{dx}(x) = \begin{cases} \dfrac{2(h-x)}{h^2}, & \text{if } 0 \leq x \leq h, \\ 0, & \text{otherwise.} \end{cases}$$

Solution to Problem 3.6. Let X be the waiting time and Y be the number of customers found. For $x < 0$, we have $F_X(x) = 0$, while for $x \geq 0$,

$$F_X(x) = \mathbf{P}(X \leq x) = \frac{1}{2}\mathbf{P}(X \leq x \,|\, Y = 0) + \frac{1}{2}\mathbf{P}(X \leq x \,|\, Y = 1).$$

Since

$$\mathbf{P}(X \leq x \,|\, Y = 0) = 1,$$

$$\mathbf{P}(X \leq x \,|\, Y = 1) = 1 - e^{-\lambda x},$$

we obtain

$$F_X(x) = \begin{cases} \frac{1}{2}(2 - e^{-\lambda x}), & \text{if } x \geq 0, \\ 0, & \text{otherwise.} \end{cases}$$

Note that the CDF has a discontinuity at $x = 0$. The random variable X is neither discrete nor continuous.

Solution to Problem 3.7. (a) We first calculate the CDF of X. For $x \in [0, r]$, we have

$$F_X(x) = \mathbf{P}(X \leq x) = \frac{\pi x^2}{\pi r^2} = \left(\frac{x}{r}\right)^2.$$

For $x < 0$, we have $F_X(x) = 0$, and for $x > r$, we have $F_X(x) = 1$. By differentiating, we obtain the PDF

$$f_X(x) = \begin{cases} \frac{2x}{r^2}, & \text{if } 0 \leq x \leq r, \\ 0, & \text{otherwise.} \end{cases}$$

We have

$$\mathbf{E}[X] = \int_0^r \frac{2x^2}{r^2}\,dx = \frac{2r}{3}.$$

Also

$$\mathbf{E}[X^2] = \int_0^r \frac{2x^3}{r^2}\,dx = \frac{r^2}{2},$$

so

$$\text{var}(X) = \mathbf{E}[X^2] - \big(\mathbf{E}[X]\big)^2 = \frac{r^2}{2} - \frac{4r^2}{9} = \frac{r^2}{18}.$$

(b) Alvin gets a positive score in the range $[1/t, \infty)$ if and only if $X \leq t$, and otherwise he gets a score of 0. Thus, for $s < 0$, the CDF of S is $F_S(s) = 0$. For $0 \leq s < 1/t$, we have

$$F_S(s) = \mathbf{P}(S \leq s) = \mathbf{P}(\text{Alvin's hit is outside the inner circle}) = 1 - \mathbf{P}(X \leq t) = 1 - \frac{t^2}{r^2}.$$

For $1/t < s$, the CDF of S is given by

$$F_S(s) = \mathbf{P}(S \leq s) = \mathbf{P}(X \leq t)\mathbf{P}(S \leq s \,|\, X \leq t) + \mathbf{P}(X > t)\mathbf{P}(S \leq s \,|\, X > t).$$

We have

$$\mathbf{P}(X \le t) = \frac{t^2}{r^2}, \qquad \mathbf{P}(X > t) = 1 - \frac{t^2}{r^2},$$

and since $S = 0$ when $X > t$,

$$\mathbf{P}(S \le s \,|\, X > t) = 1.$$

Furthermore,

$$\mathbf{P}(S \le s \,|\, X \le t) = \mathbf{P}(1/X \le s \,|\, X \le t) = \frac{\mathbf{P}(1/s \le X \le t)}{\mathbf{P}(X \le t)} = \frac{\dfrac{\pi t^2 - \pi(1/s)^2}{\pi r^2}}{\dfrac{\pi t^2}{\pi r^2}} = 1 - \frac{1}{s^2 t^2}.$$

Combining the above equations, we obtain

$$\mathbf{P}(S \le s) = \frac{t^2}{r^2}\left(1 - \frac{1}{s^2 t^2}\right) + 1 - \frac{t^2}{r^2} = 1 - \frac{1}{s^2 r^2}.$$

Collecting the results of the preceding calculations, the CDF of S is

$$F_S(s) = \begin{cases} 0, & \text{if } s < 0, \\ 1 - \dfrac{t^2}{r^2}, & \text{if } 0 \le s < 1/t, \\ 1 - \dfrac{1}{s^2 r^2}, & \text{if } 1/t \le s. \end{cases}$$

Because F_S has a discontinuity at $s = 0$, the random variable S is not continuous.

Solution to Problem 3.8. (a) By the total probability theorem, we have

$$F_X(x) = \mathbf{P}(X \le x) = p\mathbf{P}(Y \le x) + (1-p)\mathbf{P}(Z \le x) = pF_Y(x) + (1-p)F_Z(x).$$

By differentiating, we obtain

$$f_X(x) = pf_Y(x) + (1-p)f_Z(x).$$

(b) Consider the random variable Y that has PDF

$$f_Y(y) = \begin{cases} \lambda e^{\lambda y}, & \text{if } y < 0 \\ 0, & \text{otherwise,} \end{cases}$$

and the random variable Z that has PDF

$$f_Z(z) = \begin{cases} \lambda e^{-\lambda z}, & \text{if } y \ge 0 \\ 0, & \text{otherwise.} \end{cases}$$

We note that the random variables $-Y$ and Z are exponential. Using the CDF of the exponential random variable, we see that the CDFs of Y and Z are given by

$$F_Y(y) = \begin{cases} e^{\lambda y}, & \text{if } y < 0, \\ 1, & \text{if } y \ge 0, \end{cases}$$

$$F_Z(z) = \begin{cases} 0, & \text{if } z < 0, \\ 1 - e^{-\lambda z}, & \text{if } z \geq 0. \end{cases}$$

We have $f_X(x) = pf_Y(x) + (1-p)f_Z(x)$, and consequently $F_X(x) = pF_Y(x) + (1-p)F_Z(x)$. It follows that

$$F_X(x) = \begin{cases} pe^{\lambda x}, & \text{if } x < 0, \\ p + (1-p)(1-e^{-\lambda x}), & \text{if } x \geq 0, \end{cases}$$
$$= \begin{cases} pe^{\lambda x}, & \text{if } x < 0, \\ 1 - (1-p)e^{-\lambda x}, & \text{if } x \geq 0. \end{cases}$$

Solution to Problem 3.11. (a) X is a standard normal, so by using the normal table, we have $\mathbf{P}(X \leq 1.5) = \Phi(1.5) = 0.9332$. Also $\mathbf{P}(X \leq -1) = 1 - \Phi(1) = 1 - 0.8413 = 0.1587$.

(b) The random variable $(Y-1)/2$ is obtained by subtracting from Y its mean (which is 1) and dividing by the standard deviation (which is 2), so the PDF of $(Y-1)/2$ is the standard normal.

(c) We have, using the normal table,

$$\begin{aligned} \mathbf{P}(-1 \leq Y \leq 1) &= \mathbf{P}\big(-1 \leq (Y-1)/2 \leq 0\big) \\ &= \mathbf{P}(-1 \leq Z \leq 0) \\ &= \mathbf{P}(0 \leq Z \leq 1) \\ &= \Phi(1) - \Phi(0) \\ &= 0.8413 - 0.5 \\ &= 0.3413, \end{aligned}$$

where Z is a standard normal random variable.

Solution to Problem 3.12. The random variable $Z = X/\sigma$ is a standard normal, so

$$\mathbf{P}(X \geq k\sigma) = \mathbf{P}(Z \geq k) = 1 - \Phi(k).$$

From the normal tables we have

$$\Phi(1) = 0.8413, \qquad \Phi(2) = 0.9772, \qquad \Phi(3) = 0.9986.$$

Thus $\mathbf{P}(X \geq \sigma) = 0.1587$, $\mathbf{P}(X \geq 2\sigma) = 0.0228$, $\mathbf{P}(X \geq 3\sigma) = 0.0014$.

We also have

$$\mathbf{P}\big(|X| \leq k\sigma\big) = \mathbf{P}\big(|Z| \leq k\big) = \Phi(k) - \mathbf{P}(Z \leq -k) = \Phi(k) - \big(1 - \Phi(k)\big) = 2\Phi(k) - 1.$$

Using the normal table values above, we obtain

$$\mathbf{P}(|X| \leq \sigma) = 0.6826, \qquad \mathbf{P}(|X| \leq 2\sigma) = 0.9544, \qquad \mathbf{P}(|X| \leq 3\sigma) = 0.9972,$$

where t is a standard normal random variable.

Solution to Problem 3.13. Let X and Y be the temperature in Celsius and Fahrenheit, respectively, which are related by $X = 5(Y - 32)/9$. Therefore, 59 degrees Fahrenheit correspond to 15 degrees Celsius. So, if Z is a standard normal random variable, we have using $\mathbf{E}[X] = \sigma_X = 10$,

$$\mathbf{P}(Y \leq 59) = \mathbf{P}(X \leq 15) = \mathbf{P}\left(Z \leq \frac{15 - \mathbf{E}[X]}{\sigma_X}\right) = \mathbf{P}(Z \leq 0.5) = \Phi(0.5).$$

From the normal tables we have $\Phi(0.5) = 0.6915$, so $\mathbf{P}(Y \leq 59) = 0.6915$.

Solution to Problem 3.15. (a) Since the area of the semicircle is $\pi r^2/2$, the joint PDF of X and Y is $f_{X,Y}(x, y) = 2/\pi r^2$, for (x, y) in the semicircle, and $f_{X,Y}(x, y) = 0$, otherwise.

(b) To find the marginal PDF of Y, we integrate the joint PDF over the range of X. For any possible value y of Y, the range of possible values of X is the interval $[-\sqrt{r^2 - y^2}, \sqrt{r^2 - y^2}]$, and we have

$$f_Y(y) = \int_{-\sqrt{r^2-y^2}}^{\sqrt{r^2-y^2}} \frac{2}{\pi r^2}\, dx = \begin{cases} \dfrac{4\sqrt{r^2 - y^2}}{\pi r^2}, & \text{if } 0 \leq y \leq r, \\ 0, & \text{otherwise.} \end{cases}$$

Thus,

$$\mathbf{E}[Y] = \frac{4}{\pi r^2} \int_0^r y\sqrt{r^2 - y^2}\, dy = \frac{4r}{3\pi},$$

where the integration is performed using the substitution $z = r^2 - y^2$.

(c) There is no need to find the marginal PDF f_Y in order to find $\mathbf{E}[Y]$. Let D denote the semicircle. We have, using polar coordinates

$$\mathbf{E}[Y] = \int\int_{(x,y)\in D} y f_{X,Y}(x, y)\, dx\, dy = \int_0^\pi \int_0^r \frac{2}{\pi r^2} s(\sin\theta) s\, ds\, d\theta = \frac{4r}{3\pi}.$$

Solution to Problem 3.16. Let A be the event that the needle will cross a horizontal line, and let B be the probability that it will cross a vertical line. From the analysis of Example 3.11, we have that

$$\mathbf{P}(A) = \frac{2l}{\pi a}, \qquad \mathbf{P}(B) = \frac{2l}{\pi b}.$$

Since at most one horizontal (or vertical) line can be crossed, the expected number of horizontal lines crossed is $\mathbf{P}(A)$ [or $\mathbf{P}(B)$, respectively]. Thus the expected number of crossed lines is

$$\mathbf{P}(A) + \mathbf{P}(B) = \frac{2l}{\pi a} + \frac{2l}{\pi b} = \frac{2l(a + b)}{\pi ab}.$$

The probability that at least one line will be crossed is

$$\mathbf{P}(A \cup B) = \mathbf{P}(A) + \mathbf{P}(B) - \mathbf{P}(A \cap B).$$

Let X (or Y) be the distance from the needle's center to the nearest horizontal (or vertical) line. Let Θ be the angle formed by the needle's axis and the horizontal lines as in Example 3.11. We have

$$\mathbf{P}(A\cap B)=\mathbf{P}\left(X\leq \frac{l\sin\Theta}{2},\, Y\leq \frac{l\cos\Theta}{2}\right).$$

We model the triple (X, Y, Θ) as uniformly distributed over the set of all (x, y, θ) that satisfy $0 \leq x \leq a/2$, $0 \leq y \leq b/2$, and $0 \leq \theta \leq \pi/2$. Hence, within this set, we have

$$f_{X,Y,\Theta}(x,y,\theta)=\frac{8}{\pi ab}.$$

The probability $\mathbf{P}(A\cap B)$ is

$$\begin{aligned}
\mathbf{P}\big(X\leq (l/2)\sin\Theta,\, Y\leq (l/2)\cos\Theta\big) &= \iint\limits_{\substack{x\leq (l/2)\sin\theta\\ y\leq (l/2)\cos\theta}} f_{X,Y,\Theta}(x,y,\theta)\,dx\,dy\,d\theta\\
&=\frac{8}{\pi ab}\int_0^{\pi/2}\int_0^{(l/2)\cos\theta}\int_0^{(l/2)\sin\theta} dx\,dy\,d\theta\\
&=\frac{2l^2}{\pi ab}\int_0^{\pi/2}\cos\theta\,\sin\theta\,d\theta\\
&=\frac{l^2}{\pi ab}.
\end{aligned}$$

Thus we have

$$\mathbf{P}(A\cup B)=\mathbf{P}(A)+\mathbf{P}(B)-\mathbf{P}(A\cap B)=\frac{2l}{\pi a}+\frac{2l}{\pi b}-\frac{l^2}{\pi ab}=\frac{l}{\pi ab}\big(2(a+b)-l\big).$$

Solution to Problem 3.18. (a) We have

$$\mathbf{E}[X]=\int_1^3\frac{x^2}{4}\,dx=\frac{x^3}{12}\Big|_1^3=\frac{27}{12}-\frac{1}{12}=\frac{26}{12}=\frac{13}{6},$$

$$\mathbf{P}(A)=\int_2^3\frac{x}{4}\,dx=\frac{x^2}{8}\Big|_2^3=\frac{9}{8}-\frac{4}{8}=\frac{5}{8}.$$

We also have

$$\begin{aligned}
f_{X|A}(x)&=\begin{cases}\dfrac{f_X(x)}{\mathbf{P}(A)}, & \text{if } x\in A,\\ 0, & \text{otherwise,}\end{cases}\\
&=\begin{cases}\dfrac{2x}{5}, & \text{if } 2\leq x\leq 3,\\ 0, & \text{otherwise,}\end{cases}
\end{aligned}$$

from which we obtain

$$\mathbf{E}[X \mid A] = \int_2^3 x \cdot \frac{2x}{5}\, dx = \frac{2x^3}{15}\Big|_2^3 = \frac{54}{15} - \frac{16}{15} = \frac{38}{15}.$$

(b) We have

$$\mathbf{E}[Y] = \mathbf{E}[X^2] = \int_1^3 \frac{x^3}{4}\, dx = 5,$$

and

$$\mathbf{E}[Y^2] = \mathbf{E}[X^4] = \int_1^3 \frac{x^5}{4}\, dx = \frac{91}{3}.$$

Thus,

$$\mathrm{var}(Y) = \mathbf{E}[Y^2] - \big(\mathbf{E}[Y]\big)^2 = \frac{91}{3} - 5^2 = \frac{16}{3}.$$

Solution to Problem 3.19. (a) We have, using the normalization property,

$$\int_1^2 cx^{-2}\, dx = 1,$$

or

$$c = \frac{1}{\displaystyle\int_1^2 x^{-2}\, dx} = 2.$$

(b) We have

$$\mathbf{P}(A) = \int_{1.5}^2 2x^{-2}\, dx = \frac{1}{3},$$

and

$$f_{X|A}(x \mid A) = \begin{cases} 6x^{-2}, & \text{if } 1.5 < x \le 2, \\ 0, & \text{otherwise.} \end{cases}$$

(c) We have

$$\mathbf{E}[Y \mid A] = \mathbf{E}[X^2 \mid A] = \int_{1.5}^2 6x^{-2}x^2\, dx = 3,$$

$$\mathbf{E}[Y^2 \mid A] = \mathbf{E}[X^4 \mid A] = \int_{1.5}^2 6x^{-2}x^4\, dx = \frac{37}{4},$$

and

$$\mathrm{var}(Y \mid A) = \frac{37}{4} - 3^2 = \frac{1}{4}.$$

Solution to Problem 3.20. The expected value in question is

$$\begin{aligned}\mathbf{E}[\text{Time}] = &\big(5 + \mathbf{E}[\text{stay of 2nd student}]\big) \cdot \mathbf{P}(\text{1st stays no more than 5 minutes}) \\ &+ \big(\mathbf{E}[\text{stay of 1st} \mid \text{stay of 1st } \ge 5] + \mathbf{E}[\text{stay of 2nd}]\big) \\ &\cdot \mathbf{P}(\text{1st stays more than 5 minutes}).\end{aligned}$$

We have $\mathbf{E}[\text{stay of 2nd student}] = 30$, and, using the memorylessness property of the exponential distribution,

$$\mathbf{E}[\text{stay of 1st} \,|\, \text{stay of 1st } \geq 5] = 5 + \mathbf{E}[\text{stay of 1st}] = 35.$$

Also

$$\mathbf{P}(\text{1st student stays no more than 5 minutes}) = 1 - e^{-5/30},$$

$$\mathbf{P}(\text{1st student stays more than 5 minutes}) = e^{-5/30}.$$

By substitution we obtain

$$\mathbf{E}[\text{Time}] = (5+30)\cdot(1-e^{-5/30}) + (35+30)\cdot e^{-5/30} = 35 + 30\cdot e^{-5/30} = 60.394.$$

Solution to Problem 3.21. (a) We have $f_Y(y) = 1/l$, for $0 \leq y \leq l$. Furthermore, given the value y of Y, the random variable X is uniform in the interval $[0, y]$. Therefore, $f_{X|Y}(x \,|\, y) = 1/y$, for $0 \leq x \leq y$. We conclude that

$$f_{X,Y}(x,y) = f_Y(y) f_{X|Y}(x \,|\, y) = \begin{cases} \dfrac{1}{l}\cdot\dfrac{1}{y}, & 0 \leq x \leq y \leq l, \\ 0, & \text{otherwise.} \end{cases}$$

(b) We have

$$f_X(x) = \int f_{X,Y}(x,y)\,dy = \int_x^l \frac{1}{ly}\,dy = \frac{1}{l}\ln(l/x), \qquad 0 \leq x \leq l.$$

(c) We have

$$\mathbf{E}[X] = \int_0^l x f_X(x)\,dx = \int_0^l \frac{x}{l}\ln(l/x)\,dx = \frac{l}{4}.$$

(d) The fraction Y/l of the stick that is left after the first break, and the further fraction X/Y of the stick that is left after the second break are independent. Furthermore, the random variables Y and X/Y are uniformly distributed over the sets $[0, l]$ and $[0, 1]$, respectively, so that $\mathbf{E}[Y] = l/2$ and $\mathbf{E}[X/Y] = 1/2$. Thus,

$$\mathbf{E}[X] = \mathbf{E}[Y]\,\mathbf{E}\left[\frac{X}{Y}\right] = \frac{l}{2}\cdot\frac{1}{2} = \frac{l}{4}.$$

Solution to Problem 3.22. Define coordinates such that the stick extends from position 0 (the left end) to position 1 (the right end). Denote the position of the first break by X and the position of the second break by Y. With method (ii), we have $X < Y$. With methods (i) and (iii), we assume that $X < Y$ and we later account for the case $Y < X$ by using symmetry.

Under the assumption $X < Y$, the three pieces have lengths X, $Y - X$, and $1 - Y$. In order that they form a triangle, the sum of the lengths of any two pieces must exceed the length of the third piece. Thus they form a triangle if

$$X < (Y - X) + (1 - Y), \qquad (Y - X) < X + (1 - Y), \qquad (1 - Y) < X + (Y - X).$$

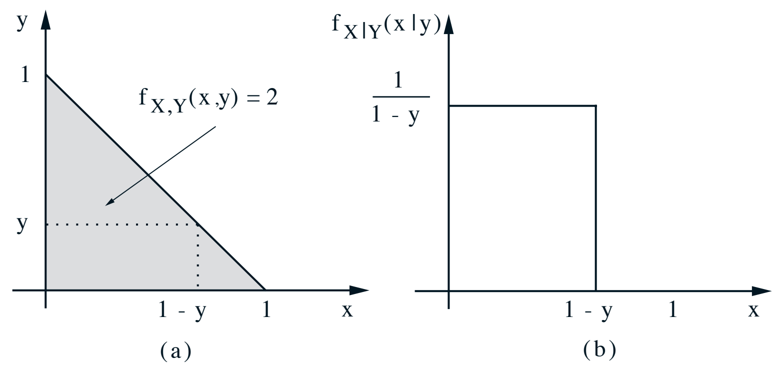

Figure 3.1: (a) The joint PDF. (b) The conditional density of X.

These conditions simplify to

$$X < 0.5, \qquad Y > 0.5, \qquad Y - X < 0.5.$$

Consider first method (i). For X and Y to satisfy these conditions, the pair (X, Y) must lie within the triangle with vertices $(0, 0.5)$, $(0.5, 0.5)$, and $(0.5, 1)$. This triangle has area $1/8$. Thus the probability of the event that the three pieces form a triangle *and* $X < Y$ is $1/8$. By symmetry, the probability of the event that the three pieces form a triangle *and* $X > Y$ is $1/8$. Since there two events are disjoint and form a partition of the event that the three pieces form a triangle, the desired probability is $1/8 + 1/8 = 1/4$.

Consider next method (ii). Since X is uniformly distributed on $[0, 1]$ and Y is uniformly distributed on $[X, 1]$, we have for $0 \le x \le y \le 1$,

$$f_{X,Y}(x, y) = f_X(x)\, f_{Y\,|\,X}(y\,|\,x) = 1 \cdot \frac{1}{1-x}.$$

The desired probability is the probability of the triangle with vertices $(0, 0.5)$, $(0.5, 0.5)$, and $(0.5, 1)$:

$$\int_0^{1/2}\int_{1/2}^{x+1/2} f_{X,Y}(x,y)dydx = \int_0^{1/2}\int_{1/2}^{x+1/2} \frac{1}{1-x}dydx = \int_0^{1/2} \frac{x}{1-x}dydx = -\frac{1}{2}+\ln 2.$$

Consider finally method (iii). Consider first the case $X < 0.5$. Then the larger piece after the first break is the piece on the right. Thus, as in method (ii), Y is uniformly distributed on $[X, 1]$ and the integral above gives the probability of a triangle being formed and $X < 0.5$. Considering also the case $X > 0.5$ doubles the probability, giving a final answer of $-1 + 2\ln 2$.

Solution to Problem 3.23. (a) The area of the triangle is $1/2$, so that $f_{X,Y}(x, y) = 1/2$, on the triangle indicated in Fig. 3.1(a), and zero everywhere else.

(b) We have

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx = \int_0^{1-y} 2\,dx = 2(1-y), \qquad 0 \le y \le 1.$$

(c) We have

$$f_{X|Y}(x\,|\,y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} = \frac{1}{1-y}, \qquad 0 \le x \le 1-y.$$

The conditional density is shown in the figure.

Intuitively, since the joint PDF is constant, the conditional PDF (which is a "slice" of the joint, at some fixed y) is also constant. Therefore, the conditional PDF must be a uniform distribution. Given that $Y = y$, X ranges from 0 to $1-y$. Therefore, for the PDF to integrate to 1, its height must be equal to $1/(1-y)$, in agreement with the figure.

(d) For $y > 1$ or $y < 0$, the conditional PDF is undefined, since these values of y are impossible. For $0 \le y < 1$, the conditional mean $\mathbf{E}[X\,|\,Y = y]$ is obtained using the uniform PDF in Fig. 3.1(b), and we have

$$\mathbf{E}[X\,|\,Y = y] = \frac{1-y}{2}, \qquad 0 \le y < 1.$$

For $y = 1$, X must be equal to 0, with certainty, so $\mathbf{E}[X\,|\,Y = 1] = 0$. Thus, the above formula is also valid when $y = 1$. The conditional expectation is undefined when y is outside $[0,1]$.

The total expectation theorem yields

$$\mathbf{E}[X] = \int_0^1 \frac{1-y}{2} f_Y(y)\,dy = \frac{1}{2} - \frac{1}{2}\int_0^1 y f_Y(y)\,dy = \frac{1-\mathbf{E}[Y]}{2}.$$

(e) Because of symmetry, we must have $\mathbf{E}[X] = \mathbf{E}[Y]$. Therefore, $\mathbf{E}[X] = \big(1-\mathbf{E}[X]\big)/2$, which yields $\mathbf{E}[X] = 1/3$.

Solution to Problem 3.24. The conditional density of X given that $Y = y$ is uniform over the interval $[0, (2-y)/2]$, and we have

$$\mathbf{E}[X\,|\,Y = y] = \frac{2-y}{4}, \qquad 0 \le y \le 2.$$

Therefore, using the total expectation theorem,

$$\mathbf{E}[X] = \int_0^2 \frac{2-y}{4} f_Y(y)\,dy = \frac{2}{4} - \frac{1}{4}\int_0^2 y f_Y(y)\,dy = \frac{2-\mathbf{E}[Y]}{4}.$$

Similarly, the conditional density of Y given that $X = x$ is uniform over the interval $[0, 2(1-x)]$, and we have

$$\mathbf{E}[Y\,|\,X = x] = 1-x, \qquad 0 \le x \le 1.$$

Therefore,

$$\mathbf{E}[Y] = \int_0^1 (1-x) f_X(x)\,dx = 1 - \mathbf{E}[X].$$

By solving the two equations above for $\mathbf{E}[X]$ and $\mathbf{E}[Y]$, we obtain

$$\mathbf{E}[X] = \frac{1}{3}, \qquad \mathbf{E}[Y] = \frac{2}{3}.$$

Solution to Problem 3.25. Let C denote the event that $X^2 + Y^2 \geq c^2$. The probability $\mathbf{P}(C)$ can be calculated using polar coordinates, as follows:

$$\begin{aligned}
\mathbf{P}(C) &= \frac{1}{2\pi\sigma^2} \int_0^{2\pi} \int_c^{\infty} r e^{-r^2/2\sigma^2}\,dr\,d\theta \\
&= \frac{1}{\sigma^2} \int_c^{\infty} r e^{-r^2/2\sigma^2}\,dr \\
&= e^{-c^2/2\sigma^2}.
\end{aligned}$$

Thus, for $(x, y) \in C$,

$$f_{X,Y|C}(x, y) = \frac{f_{X,Y}(x, y)}{\mathbf{P}(C)} = \frac{1}{2\pi\sigma^2} e^{-\frac{1}{2\sigma^2}(x^2 + y^2 - c^2)}.$$

Solution to Problem 3.34. (a) Let A be the event that the first coin toss resulted in heads. To calculate the probability $\mathbf{P}(A)$, we use the continuous version of the total probability theorem:

$$\mathbf{P}(A) = \int_0^1 \mathbf{P}(A \,|\, P = p) f_P(p)\,dp = \int_0^1 p^2 e^p\,dp,$$

which after some calculation yields

$$\mathbf{P}(A) = e - 2.$$

(b) Using Bayes' rule,

$$\begin{aligned}
f_{P|A}(p) &= \frac{\mathbf{P}(A|P = p) f_P(p)}{\mathbf{P}(A)} \\
&= \begin{cases} \dfrac{p^2 e^p}{e-2}, & 0 \leq p \leq 1, \\ 0, & \text{otherwise.} \end{cases}
\end{aligned}$$

(c) Let B be the event that the second toss resulted in heads. We have

$$\begin{aligned}
\mathbf{P}(B \,|\, A) &= \int_0^1 \mathbf{P}(B \,|\, P = p, A) f_{P|A}(p)\,dp \\
&= \int_0^1 \mathbf{P}(B \,|\, P = p) f_{P|A}(p)\,dp \\
&= \frac{1}{e-2} \int_0^1 p^3 e^p\,dp.
\end{aligned}$$

After some calculation, this yields

$$\mathbf{P}(B \mid A) = \frac{1}{e-2} \cdot (6-2e) = \frac{0.564}{0.718} \approx 0.786.$$