# CHAPTER 4

**Solution to Problem 4.1.** Let $Y = \sqrt{|X|}$. We have, for $0 \le y \le 1$,

$$F_Y(y) = \mathbf{P}(Y \le y) = \mathbf{P}(\sqrt{|X|} \le y) = \mathbf{P}(-y^2 \le X \le y^2) = y^2,$$

and therefore by differentiation,

$$f_Y(y) = 2y, \qquad \text{for } 0 \le y \le 1.$$

Let $Y = -\ln|X|$. We have, for $y \ge 0$,

$$F_Y(y) = \mathbf{P}(Y \le y) = \mathbf{P}(\ln|X| \ge -y) = \mathbf{P}(X \ge e^{-y}) + \mathbf{P}(X \le -e^{-y}) = 1 - e^{-y},$$

and therefore by differentiation

$$f_Y(y) = e^{-y}, \qquad \text{for } y \ge 0,$$

so $Y$ is an exponential random variable with parameter 1. This exercise provides a method for simulating an exponential random variable using a sample of a uniform random variable.

**Solution to Problem 4.2.** Let $Y = e^X$. We first find the CDF of $Y$, and then take the derivative to find its PDF. We have

$$\mathbf{P}(Y \le y) = \mathbf{P}(e^X \le y) = \begin{cases} \mathbf{P}(X \le \ln y), & \text{if } y > 0, \\ 0, & \text{otherwise.} \end{cases}$$

Therefore,

$$f_Y(y) = \begin{cases} \dfrac{d}{dx} F_X(\ln y), & \text{if } y > 0, \\ 0, & \text{otherwise,} \end{cases}$$
$$= \begin{cases} \dfrac{1}{y} f_X(\ln y), & \text{if } y > 0, \\ 0, & \text{otherwise.} \end{cases}$$

When $X$ is uniform on $[0, 1]$, the answer simplifies to

$$f_Y(y) = \begin{cases} \dfrac{1}{y}, & \text{if } 0 < y \le e, \\ 0, & \text{otherwise.} \end{cases}$$

**Solution to Problem 4.3.** Let $Y = |X|^{1/3}$. We have

$$F_Y(y) = \mathbf{P}(Y \le y) = \mathbf{P}\big(|X|^{1/3} \le y\big) = \mathbf{P}\big(-y^3 \le X \le y^3\big) = F_X(y^3) - F_X(-y^3),$$

and therefore, by differentiating,

$$f_Y(y) = 3y^2 f_X(y^3) + 3y^2 f_X(-y^3), \qquad \text{for } y > 0.$$

Let $Y = |X|^{1/4}$. We have

$$F_Y(y) = \mathbf{P}(Y \le y) = \mathbf{P}\big(|X|^{1/4} \le y\big) = \mathbf{P}(-y^4 \le X \le y^4) = F_X(y^4) - F_X(-y^4),$$

and therefore, by differentiating,

$$f_Y(y) = 4y^3 f_X(y^4) + 4y^3 f_X(-y^4), \qquad \text{for } y > 0.$$

**Solution to Problem 4.4.** We have

$$F_Y(y) = \begin{cases} 0, & \text{if } y \le 0, \\ \mathbf{P}(5 - y \le X \le 5) + \mathbf{P}(20 - y \le X \le 20), & \text{if } 0 \le y \le 5, \\ \mathbf{P}(20 - y \le X \le 20), & \text{if } 5 < y \le 15, \\ 1, & \text{if } y > 15. \end{cases}$$

Using the CDF of $X$, we have

$$\mathbf{P}(5 - y \le X \le 5) = F_X(5) - F_X(5 - y),$$

$$\mathbf{P}(20 - y \le X \le 20) = F_X(20) - F_X(20 - y).$$

Thus,

$$F_Y(y) = \begin{cases} 0, & \text{if } y \le 0, \\ F_X(5) - F_X(5 - y) + F_X(20) - F_X(20 - y), & \text{if } 0 \le y \le 5, \\ F_X(20) - F_X(20 - y), & \text{if } 5 < y \le 15, \\ 1, & \text{if } y > 15. \end{cases}$$

Differentiating, we obtain

$$f_Y(y) = \begin{cases} f_X(5 - y) + f_X(20 - y), & \text{if } 0 \le y \le 5, \\ f_X(20 - y), & \text{if } 5 < y \le 15, \\ 0, & \text{otherwise,} \end{cases}$$

consistent with the result of Example 3.14.

**Solution to Problem 4.5.** Let $Z = |X - Y|$. We have

$$F_Z(z) = P\big(|X - Y| \le z\big) = 1 - (1 - z)^2.$$

(To see this, draw the event of interest as a subset of the unit square and calculate its area.) Taking derivatives, the desired PDF is

$$f_Z(z) = \begin{cases} 2(1 - z), & \text{if } 0 \le z \le 1, \\ 0, & \text{otherwise.} \end{cases}$$

**Solution to Problem 4.6.** Let $Z = |X - Y|$. To find the CDF, we integrate the joint PDF of $X$ and $Y$ over the region where $|X - Y| \leq z$ for a given $z$. In the case where $z \leq 0$ or $z \geq 1$, the CDF is 0 and 1, respectively. In the case where $0 < z < 1$, we have

$$F_Z(z) = \mathbf{P}(X - Y \leq z,\, X \geq Y) + \mathbf{P}(Y - X \leq z,\, X < Y).$$

The events $\{X - Y \leq z,\, X \geq Y\}$ and $\{Y - X \leq z,\, X < Y\}$ can be identified with subsets of the given triangle. After some calculation using triangle geometry, the areas of these subsets can be verified to be $z/2 + z^2/4$ and $1/4 - (1-z)^2/4$, respectively. Therefore, since $f_{X,Y}(x,y) = 1$ for all $(x,y)$ in the given triangle,

$$F_Z(z) = \left(\frac{z}{2} + \frac{z^2}{4}\right) + \left(\frac{1}{4} - \frac{(1-z)^2}{4}\right) = z.$$

Thus,

$$F_Z(z) = \begin{cases} 0, & \text{if } z \leq 0, \\ z, & \text{if } 0 < z < 1, \\ 1, & \text{if } z \geq 1. \end{cases}$$

By taking the derivative with respect to $z$, we obtain

$$f_Z(z) = \begin{cases} 1, & \text{if } 0 \leq z \leq 1, \\ 0, & \text{otherwise.} \end{cases}$$

**Solution to Problem 4.7.** Let $X$ and $Y$ be the two points, and let $Z = \max\{X, Y\}$. For any $t \in [0, 1]$, we have

$$\mathbf{P}(Z \leq t) = \mathbf{P}(X \leq t)\mathbf{P}(Y \leq t) = t^2,$$

and by differentiating, the corresponding PDF is

$$f_Z(z) = \begin{cases} 0, & \text{if } z \leq 0, \\ 2z, & \text{if } 0 \leq z \leq 1, \\ 0, & \text{if } z \geq 1. \end{cases}$$

Thus, we have

$$\mathbf{E}[Z] = \int_{-\infty}^{\infty} z f_Z(z) dz = \int_0^1 2z^2 dz = \frac{2}{3}.$$

The distance of the largest of the two points to the right endpoint is $1 - Z$, and its expected value is $1 - \mathbf{E}[Z] = 1/3$. A symmetric argument shows that the distance of the smallest of the two points to the left endpoint is also $1/3$. Therefore, the expected distance between the two points must also be $1/3$.

**Solution to Problem 4.8.** Note that $f_X(x)$ and $f_Y(z - x)$ are nonzero only when $x \geq 0$ and $x \leq z$, respectively. Thus, in the convolution formula, we only need to integrate for $x$ ranging from 0 to $z$:

$$f_Z(z) = \int_{-\infty}^{\infty} f_X(x) f_Y(z - x)\, dx = \int_0^z \lambda e^{-\lambda x} \lambda e^{-\lambda(z-x)}\, dx = \lambda^2 e^{-z} \int_0^z dx = \lambda^2 z e^{-\lambda z}.$$

**Solution to Problem 4.9.** Let $Z = X - Y$. We will first calculate the CDF $F_Z(z)$ by considering separately the cases $z \geq 0$ and $z < 0$. For $z \geq 0$, we have (see the left side of Fig. 4.6)

$$\begin{aligned} F_Z(z) &= \mathbf{P}(X - Y \leq z) \\ &= 1 - \mathbf{P}(X - Y > z) \\ &= 1 - \int_0^\infty \left( \int_{z+y}^\infty f_{X,Y}(x,y)\, dx \right) dy \\ &= 1 - \int_0^\infty \mu e^{-\mu y} \left( \int_{z+y}^\infty \lambda e^{-\lambda x}\, dx \right) dy \\ &= 1 - \int_0^\infty \mu e^{-\mu y} e^{-\lambda(z+y)}\, dy \\ &= 1 - e^{-\lambda z} \int_0^\infty \mu e^{-(\lambda+\mu)y}\, dy \\ &= 1 - \frac{\mu}{\lambda+\mu} e^{-\lambda z}. \end{aligned}$$

For the case $z < 0$, we have using the preceding calculation

$$F_Z(z) = 1 - F_Z(-z) = 1 - \left( 1 - \frac{\lambda}{\lambda+\mu} e^{-\mu(-z)} \right) = \frac{\lambda}{\lambda+\mu} e^{\mu z}.$$

Combining the two cases $z \geq 0$ and $z < 0$, we obtain

$$F_Z(z) = \begin{cases} 1 - \dfrac{\mu}{\lambda+\mu} e^{-\lambda z}, & \text{if } z \geq 0, \\ \dfrac{\lambda}{\lambda+\mu} e^{\mu z}, & \text{if } z < 0. \end{cases}$$

The PDF of $Z$ is obtained by differentiating its CDF. We have

$$f_Z(z) = \begin{cases} \dfrac{\lambda\mu}{\lambda+\mu} e^{-\lambda z}, & \text{if } z \geq 0, \\ \dfrac{\lambda\mu}{\lambda+\mu} e^{\mu z}, & \text{if } z < 0. \end{cases}$$

For an alternative solution, fix some $z \geq 0$ and note that $f_Y(x - z)$ is nonzero only when $x \geq z$. Thus,

$$\begin{aligned} f_{X-Y}(z) &= \int_{-\infty}^\infty f_X(x) f_Y(x-z)\, dx \\ &= \int_z^\infty \lambda e^{-\lambda x} \mu e^{-\mu(x-z)}\, dx \\ &= \lambda\mu e^{\lambda z} \int_z^\infty e^{-(\lambda+\mu)x}\, dx \\ &= \lambda\mu e^{\lambda z} \frac{1}{\lambda+\mu} e^{-(\lambda+\mu)z} \\ &= \frac{\lambda\mu}{\lambda+\mu} e^{-\mu z}, \end{aligned}$$

in agreement with the earlier answer. The solution for the case $z < 0$ is obtained with a similar calculation.

**Solution to Problem 4.10.** We first note that the range of possible values of $Z$ are the integers from the range $[1, 5]$. Thus we have

$$p_Z(z) = 0, \quad \text{if } z \neq 1, 2, 3, 4, 5.$$

We calculate $p_Z(z)$ for each of the values $z = 1, 2, 3, 4, 5$, using the convolution formula. We have

$$p_Z(1) = \sum_x p_X(x) p_Y(1 - x) = p_X(1) p_Y(0) = \frac{1}{3} \cdot \frac{1}{2} = \frac{1}{6},$$

where the second equality above is based on the fact that for $x \neq 1$ either $p_X(x)$ or $p_Y(1 - x)$ (or both) is zero. Similarly, we obtain

$$p_Z(2) = p_X(1) p_Y(1) + p_X(2) p_Y(0) = \frac{1}{3} \cdot \frac{1}{3} + \frac{1}{3} \cdot \frac{1}{2} = \frac{5}{18},$$

$$p_Z(3) = p_X(1) p_Y(2) + p_X(2) p_Y(1) + p_X(3) p_Y(0) = \frac{1}{3} \cdot \frac{1}{6} + \frac{1}{3} \cdot \frac{1}{3} + \frac{1}{3} \cdot \frac{1}{2} = \frac{1}{3},$$

$$p_Z(4) = p_X(2) p_Y(2) + p_X(3) p_Y(1) = \frac{1}{3} \cdot \frac{1}{6} + \frac{1}{3} \cdot \frac{1}{3} = \frac{1}{6},$$

$$p_Z(5) = p_X(3) p_Y(2) = \frac{1}{3} \cdot \frac{1}{6} = \frac{1}{18}.$$

**Solution to Problem 4.11.** The convolution of two Poisson PMFs is of the form

$$\sum_{i=0}^{k} \frac{\lambda^i e^{-\lambda}}{i!} \cdot \frac{\mu^{k-i} e^{-\mu}}{(k-i)!} = e^{-(\lambda+\mu)} \sum_{i=0}^{k} \frac{\lambda^i \mu^{k-i}}{i!\,(k-i)!}.$$

We have

$$(\lambda + \mu)^k = \sum_{i=0}^{k} \binom{k}{i} \lambda^i \mu^{k-i} = \sum_{i=0}^{k} \frac{k!}{i!\,(k-i)!} \lambda^i \mu^{k-i}.$$

Thus, the desired PMF is

$$\frac{e^{-(\lambda+\mu)}}{k!} \sum_{i=0}^{k} \frac{k!\,\lambda^i \mu^{k-i}}{i!\,(k-i)!} = \frac{e^{-(\lambda+\mu)}}{k!} (\lambda + \mu)^k,$$

which is a Poisson PMF with mean $\lambda + \mu$.

**Solution to Problem 4.12.** Let $V = X + Y$. As in Example 4.10, the PDF of $V$ is

$$f_V(v) = \begin{cases} v, & 0 \leq v \leq 1, \\ 2 - v, & 1 \leq v \leq 2, \\ 0, & \text{otherwise.} \end{cases}$$

Let $W = X + Y + Z = V + Z$. We convolve the PDFs $f_V$ and $f_Z$, to obtain

$$f_W(w) = \int f_V(v) f_Z(w - v)\, dv.$$

We first need to determine the limits of the integration. Since $f_V(v) = 0$ outside the range $0 \le v \le 2$, and $f_W(w - v) = 0$ outside the range $0 \le w - v \le 1$, we see that the integrand can be nonzero only if

$$0 \le v \le 2, \qquad \text{and} \qquad w - 1 \le v \le w.$$

We consider three separate cases. If $w \le 1$, we have

$$f_W(w) = \int_0^w f_V(v) f_Z(w - v)\, dv = \int_0^w v\, dv = \frac{w^2}{2}.$$

If $1 \le w \le 2$, we have

$$\begin{aligned} f_W(w) &= \int_{w-1}^{w} f_V(v) f_Z(w - v)\, dv \\ &= \int_{w-1}^{1} v\, dv + \int_1^w (2 - v)\, dv \\ &= \frac{1}{2} - \frac{(w-1)^2}{2} - \frac{(w-2)^2}{2} + \frac{1}{2}. \end{aligned}$$

Finally, if $2 \le w \le 3$, we have

$$f_W(w) = \int_{w-1}^{2} f_V(v) f_Z(w - v)\, dv = \int_{w-1}^{2} (2 - v)\, dv = \frac{(3-w)^2}{2}.$$

To summarize,

$$f_W(w) = \begin{cases} w^2/2, & 0 \le w \le 1, \\ 1 - (w-1)^2/2 - (2-w)^2/2, & 1 \le w \le 2, \\ (3-w)^2/2, & 2 \le w \le 3, \\ 0, & \text{otherwise.} \end{cases}$$

**Solution to Problem 4.13.** We have $X - Y = X + Z - (a + b)$, where $Z = a + b - Y$ is distributed identically with $X$ and $Y$. Thus, the PDF of $X + Z$ is the same as the PDF of $X + Y$, and the PDF of $X - Y$ is obtained by shifting the PDF of $X + Y$ to the left by $a + b$.

**Solution to Problem 4.14.** For all $z \ge 0$, we have, using the independence of $X$ and $Y$, and the form of the exponential CDF,

$$\begin{aligned} F_Z(z) &= \mathbf{P}\big(\min\{X, Y\} \le z\big) \\ &= 1 - \mathbf{P}\big(\min\{X, Y\} > z\big) \\ &= 1 - \mathbf{P}(X > z,\, Y > z) \\ &= 1 - \mathbf{P}(X > z)\mathbf{P}(Y > z) \\ &= 1 - e^{-\lambda z} e^{-\mu z} \\ &= 1 - e^{-(\lambda + \mu) z}. \end{aligned}$$

This is recognized as the exponential CDF with parameter $\lambda + \mu$. Thus, the minimum of two independent exponentials with parameters $\lambda$ and $\mu$ is an exponential with parameter $\lambda + \mu$.

**Solution to Problem 4.17.** Because the covariance remains unchanged when we add a constant to a random variable, we can assume without loss of generality that $X$ and $Y$ have zero mean. We then have

$$\text{cov}(X - Y, X + Y) = \mathbf{E}\big[(X - Y)(X + Y)\big] = \mathbf{E}[X^2] - \mathbf{E}[Y^2] = \text{var}(X) - \text{var}(Y) = 0,$$

since $X$ and $Y$ were assumed to have the same variance.

**Solution to Problem 4.18.** We have

$$\text{cov}(R, S) = \mathbf{E}[RS] - \mathbf{E}[R]\mathbf{E}[S] = \mathbf{E}[WX + WY + X^2 + XY] = \mathbf{E}[X^2] = 1,$$

and

$$\text{var}(R) = \text{var}(S) = 2,$$

so

$$\rho(R, S) = \frac{\text{cov}(R, S)}{\sqrt{\text{var}(R)\text{var}(S)}} = \frac{1}{2}.$$

We also have

$$\text{cov}(R, T) = \mathbf{E}[RT] - \mathbf{E}[R]\mathbf{E}[T] = \mathbf{E}[WY + WZ + XY + XZ] = 0,$$

so that

$$\rho(R, T) = 0.$$

**Solution to Problem 4.19.** To compute the correlation coefficient

$$\rho(X, Y) = \frac{\text{cov}(X, Y)}{\sigma_X \sigma_Y},$$

we first compute the covariance:

$$\begin{aligned}
\text{cov}(X, Y) &= \mathbf{E}[XY] - \mathbf{E}[X]\mathbf{E}[Y] \\
&= \mathbf{E}[aX + bX^2 + cX^3] - \mathbf{E}[X]\mathbf{E}[Y] \\
&= a\mathbf{E}[X] + b\mathbf{E}[X^2] + c\mathbf{E}[X^3] \\
&= b.
\end{aligned}$$

We also have

$$\begin{aligned}
\text{var}(Y) &= \text{var}(a + bX + cX^2) \\
&= \mathbf{E}\big[(a + bX + cX^2)^2\big] - \big(\mathbf{E}[a + bX + cX^2]\big)^2 \\
&= (a^2 + 2ac + b^2 + 3c^2) - (a^2 + c^2 + 2ac) \\
&= b^2 + 2c^2,
\end{aligned}$$

and therefore, using the fact $\text{var}(X) = 1$,

$$\rho(X, Y) = \frac{b}{\sqrt{b^2 + 2c^2}}.$$

**Solution to Problem 4.22.** If the gambler's fortune at the beginning of a round is $a$, the gambler bets $a(2p-1)$. He therefore gains $a(2p-1)$ with probability $p$, and loses $a(2p-1)$ with probability $1-p$. Thus, his expected fortune at the end of a round is

$$a\Big(1 + p(2p-1) - (1-p)(2p-1)\Big) = a\Big(1 + (2p-1)^2\Big).$$

Let $X_k$ be the fortune after the $k$th round. Using the preceding calculation, we have

$$\mathbf{E}[X_{k+1} \,|\, X_k] = \Big(1 + (2p-1)^2\Big) X_k.$$

Using the law of iterated expectations, we obtain

$$\mathbf{E}[X_{k+1}] = \Big(1 + (2p-1)^2\Big) \mathbf{E}[X_k],$$

and

$$\mathbf{E}[X_1] = \Big(1 + (2p-1)^2\Big) x.$$

We conclude that

$$\mathbf{E}[X_n] = \Big(1 + (2p-1)^2\Big)^n x.$$

**Solution to Problem 4.23.** (a) Let $W$ be the number of hours that Nat waits. We have

$$\mathbf{E}[X] = \mathbf{P}(0 \le X \le 1)\mathbf{E}[W \,|\, 0 \le X \le 1] + \mathbf{P}(X > 1)\mathbf{E}[W \,|\, X > 1].$$

Since $W > 0$ only if $X > 1$, we have

$$\mathbf{E}[W] = \mathbf{P}(X > 1)\mathbf{E}[W \,|\, X > 1] = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}.$$

(b) Let $D$ be the duration of a date. We have $\mathbf{E}[D \,|\, 0 \le X \le 1] = 3$. Furthermore, when $X > 1$, the conditional expectation of $D$ given $X$ is $(3 - X)/2$. Hence, using the law of iterated expectations,

$$\mathbf{E}[D \,|\, X > 1] = \mathbf{E}\Big[\mathbf{E}[D \,|\, X] \;\,\big|\, X > 1\Big] = \mathbf{E}\left[\frac{3 - X}{2} \;\Big|\; X > 1\right].$$

Therefore,

$$\begin{aligned}
\mathbf{E}[D] &= \mathbf{P}(0 \le X \le 1)\mathbf{E}[D \,|\, 0 \le X \le 1] + \mathbf{P}(X > 1)\mathbf{E}[D \,|\, X > 1] \\
&= \frac{1}{2} \cdot 3 + \frac{1}{2} \cdot \mathbf{E}\left[\frac{3 - X}{2} \;\Big|\; X > 1\right] \\
&= \frac{3}{2} + \frac{1}{2}\left(\frac{3}{2} - \frac{\mathbf{E}[X \,|\, X > 1]}{2}\right) \\
&= \frac{3}{2} + \frac{1}{2}\left(\frac{3}{2} - \frac{3/2}{2}\right) \\
&= \frac{15}{8}.
\end{aligned}$$

(c) The probability that Pat will be late by more than 45 minutes is 1/8. The number of dates before breaking up is the sum of two geometrically distributed random variables with parameter 1/8, and its expected value is $2 \cdot 8 = 16$.

**Solution to Problem 4.24.** (a) Consider the following two random variables:

$X$ = amount of time the professor devotes to his task [exponentially distributed with parameter $\lambda(y) = 1/(5-y)$];

$Y$ = length of time between 9 a.m. and his arrival (uniformly distributed between 0 and 4).

Note that $\mathbf{E}[Y] = 2$. We have

$$\mathbf{E}[X \mid Y = y] = \frac{1}{\lambda(y)} = 5 - y,$$

which implies that

$$\mathbf{E}[X \mid Y] = 5 - Y,$$

and

$$\mathbf{E}[X] = \mathbf{E}\big[\mathbf{E}[X \mid Y]\big] = \mathbf{E}[5 - Y] = 5 - \mathbf{E}[Y] = 5 - 2 = 3.$$

(b) Let $Z$ be the length of time from 9 a.m. until the professor completes the task. Then,

$$Z = X + Y.$$

We already know from part (a) that $\mathbf{E}[X] = 3$ and $\mathbf{E}[Y] = 2$, so that

$$\mathbf{E}[Z] = \mathbf{E}[X] + \mathbf{E}[Y] = 3 + 2 = 5.$$

Thus the expected time that the professor leaves his office is 5 hours after 9 a.m.

(c) We define the following random variables:

$W$ = length of time between 9 a.m. and arrival of the Ph.D. student (uniformly distributed between 9 a.m. and 5 p.m.).

$R$ = amount of time the student will spend with the professor, if he finds the professor (uniformly distributed between 0 and 1 hour).

$T$ = amount of time the professor will spend with the student.

Let also $F$ be the event that the student finds the professor.

To find $\mathbf{E}[T]$, we write

$$\mathbf{E}[T] = \mathbf{P}(F)\mathbf{E}[T \mid F] + \mathbf{P}(F^c)\mathbf{E}[T \mid F^c]$$

Using the problem data,

$$\mathbf{E}[T \mid F] = \mathbf{E}[R] = \frac{1}{2}$$

(this is the expected value of a uniformly distribution ranging from 0 to 1),

$$\mathbf{E}[T \mid F^c] = 0$$

(since the student leaves if he does not find the professor). We have

$$\mathbf{E}[T] = \mathbf{E}[T \mid F]\mathbf{P}(F) = \frac{1}{2}\mathbf{P}(F),$$

so we need to find $\mathbf{P}(F)$.

In order for the student to find the professor, his arrival should be between the arrival and the departure of the professor. Thus

$$\mathbf{P}(F) = \mathbf{P}(Y \le W \le X + Y).$$

We have that $W$ can be between 0 (9 a.m.) and 8 (5 p.m.), but $X + Y$ can be any value greater than 0. In particular, it may happen that the sum is greater than the upper bound for $W$. We write

$$\mathbf{P}(F) = \mathbf{P}(Y \le W \le X + Y) = 1 - \big(\mathbf{P}(W < Y) + \mathbf{P}(W > X + Y)\big)$$

We have

$$\mathbf{P}(W < Y) = \int_0^4 \frac{1}{4} \int_0^y \frac{1}{8}\, dw\, dy = \frac{1}{4}$$

and

$$\begin{aligned}
\mathbf{P}(W > X + Y) &= \int_0^4 \mathbf{P}(W > X + Y \mid Y = y) f_Y(y)\, dy \\
&= \int_0^4 \mathbf{P}(X < W - Y \mid Y = y) f_Y(y)\, dy \\
&= \int_0^4 \int_y^8 F_{X|Y}(w - y) f_W(w) f_Y(y)\, dw\, dy \\
&= \int_0^4 \frac{1}{4} \int_y^8 \frac{1}{8} \int_0^{w-y} \frac{1}{5-y} e^{-\frac{x}{5-y}}\, dx\, dw\, dy \\
&= \frac{12}{32} + \frac{1}{32} \int_0^4 (5 - y) e^{-\frac{8-y}{5-y}}\, dy.
\end{aligned}$$

Integrating numerically, we have

$$\int_0^4 (5 - y) e^{-\frac{8-y}{5-y}}\, dy = 1.7584.$$

Thus,

$$\mathbf{P}(Y \le W \le X + Y) = 1 - \big(\mathbf{P}(W < Y) + \mathbf{P}(W > X + Y)\big) = 1 - 0.68 = 0.32.$$

The expected amount of time the professor will spend with the student is then

$$\mathbf{E}[T] = \frac{1}{2}\mathbf{P}(F) = \frac{1}{2}\, 0.32 = 0.16 = 9.6 \text{ mins.}$$

Next, we want to find the expected time the professor will leave his office. Let $Z$ be the length of time measured from 9 a.m. until he leaves his office. If the professor doesn't spend any time with the student, then $Z$ will be equal to $X+Y$. On the other hand, if the professor is interrupted by the student, then the length of time will be equal to $X+Y+R$. This is because the professor will spend the same amount of total time on the task regardless of whether he is interrupted by the student. Therefore,

$$\mathbf{E}[Z] = \mathbf{P}(F)\mathbf{E}[Z \mid F] + \mathbf{P}(F^c)\mathbf{E}[Z \mid F^c] = \mathbf{P}(F)\mathbf{E}[X+Y+R] + \mathbf{P}(F^c)\mathbf{E}[X+Y].$$

Using the results of the earlier calculations,

$$\mathbf{E}[X+Y] = 5,$$

$$\mathbf{E}[X+Y+R] = \mathbf{E}[X+Y] + \mathbf{E}[R] = 5 + \frac{1}{2} = \frac{11}{2}.$$

Therefore,

$$\mathbf{E}[Z] = 0.68 \cdot 5 + 0.32 \cdot \frac{11}{2} = 5.16.$$

Thus the expected time the professor will leave his office is 5.16 hours after 9 a.m.

**Solution to Problem 4.29.** The transform is given by

$$M(s) = \mathbf{E}[e^{sX}] = \frac{1}{2}e^s + \frac{1}{4}e^{2s} + \frac{1}{4}e^{3s}.$$

We have

$$\mathbf{E}[X] = \frac{d}{ds}M(s)\bigg|_{s=0} = \frac{1}{2} + \frac{2}{4} + \frac{3}{4} = \frac{7}{4},$$

$$\mathbf{E}[X^2] = \frac{d^2}{ds^2}M(s)\bigg|_{s=0} = \frac{1}{2} + \frac{4}{4} + \frac{9}{4} = \frac{15}{4},$$

$$\mathbf{E}[X^3] = \frac{d^3}{ds^3}M(s)\bigg|_{s=0} = \frac{1}{2} + \frac{8}{4} + \frac{27}{4} = \frac{37}{4}.$$

**Solution to Problem 4.30.** The transform associated with $X$ is

$$M_X(s) = e^{s^2/2}.$$

By taking derivatives with respect to $s$, we find that

$$\mathbf{E}[X] = 0, \quad \mathbf{E}[X^2] = 1, \quad \mathbf{E}[X^3] = 0, \quad \mathbf{E}[X^4] = 3.$$

**Solution to Problem 4.31.** The transform is

$$M(s) = \frac{\lambda}{\lambda - s}.$$

Thus,

$$\frac{d}{ds}M(s) = \frac{\lambda}{(\lambda - s)^2}, \qquad \frac{d^2}{ds^2}M(s) = \frac{2\lambda}{(\lambda - s)^3}, \qquad \frac{d^3}{ds^3}M(s) = \frac{6\lambda}{(\lambda - s)^4},$$

$$\frac{d^4}{ds^4}M(s) = \frac{24\lambda}{(\lambda - s)^5}, \qquad \frac{d^5}{ds^5}M(s) = \frac{120\lambda}{(\lambda - s)^6}.$$

By setting $s = 0$, we obtain

$$\mathbf{E}[X^3] = \frac{6}{\lambda^3}, \qquad \mathbf{E}[X^4] = \frac{24}{\lambda^4}, \qquad \mathbf{E}[X^5] = \frac{120}{\lambda^5}.$$

**Solution to Problem 4.32.** (a) We must have $M(0) = 1$. Only the first option satisfies this requirement.

(b) We have

$$\mathbf{P}(X = 0) = \lim_{s \to -\infty} M(s) = e^{2(e^{-1}-1)} \approx 0.2825.$$

**Solution to Problem 4.33.** We recognize this transform as corresponding to the following mixture of exponential PDFs:

$$f_X(x) = \begin{cases} \frac{1}{3} \cdot 2e^{-2x} + \frac{2}{3} \cdot 3e^{-3x}, & \text{for } x \geq 0, \\ 0, & \text{otherwise.} \end{cases}$$

By the inversion theorem, this must be the desired PDF.

**Solution to Problem 4.34.** For $i = 1, 2, 3$, let $X_i$, $i = 1, 2, 3$, be a Bernoulli random variable that takes the value 1 if the $i$th player is successful. We have $X = X_1 + X_2 + X_3$. Let $q_i = 1 - p_i$. Convolution of the PMFs of $X_1$ and $X_2$ yields the PMF of $Z = X_1 + X_2$:

$$p_Z(z) = \begin{cases} q_1 q_2, & \text{if } z = 0, \\ q_1 p_2 + p_1 q_2, & \text{if } z = 1, \\ p_1 p_2, & \text{if } z = 2, \\ 0, & \text{otherwise.} \end{cases}$$

Convolution of the PMFs of $Z$ and $X_3$ yields the PMF of $X = X_1 + X_2 + X_3$:

$$p_X(x) = \begin{cases} q_1 q_2 q_3, & \text{if } x = 0, \\ p_1 q_2 q_3 + q_1 p_2 q_3 + q_1 q_2 p_3, & \text{if } x = 1, \\ q_1 p_2 p_3 + p_1 q_2 p_3 + p_1 p_2 q_3, & \text{if } x = 2, \\ p_1 p_2 p_3, & \text{if } x = 3, \\ 0, & \text{otherwise.} \end{cases}$$

The transform associated with $X$ is the product of the transforms associated with $X_i$, $i = 1, 2, 3$. We have

$$M_X(s) = (q_1 + p_1 e^s)(q_2 + p_2 e^s)(q_3 + p_3 e^s).$$

By carrying out the multiplications above, and by examining the coefficients of the terms $e^{ks}$, we obtain the probabilities $\mathbf{P}(X = k)$. These probabilities are seen to coincide with the ones computed by convolution.

**Solution to Problem 4.35.** We first find $c$ by using the equation

$$1 = M_X(0) = c \cdot \frac{3+4+2}{3-1},$$

so that $c = 2/9$. We then obtain

$$\mathbf{E}[X] = \frac{dM_X}{ds}(s)\Big|_{s=0} = \frac{2}{9} \cdot \frac{(3-e^s)(8e^{2s}+6e^{3s}) + e^s(3+4e^{2s}+2e^{3s})}{(3-e^s)^2}\Big|_{s=0} = \frac{37}{18}.$$

We now use the identity

$$\frac{1}{3-e^s} = \frac{1}{3} \cdot \frac{1}{1-e^s/3} = \frac{1}{3}\left(1 + \frac{e^s}{3} + \frac{e^{2s}}{9} + \cdots\right),$$

which is valid as long as $s$ is small enough so that $e^s < 3$. It follows that

$$M_X(s) = \frac{2}{9} \cdot \frac{1}{3} \cdot (3 + 4e^{2s} + 2e^{3s}) \cdot \left(1 + \frac{e^s}{3} + \frac{e^{2s}}{9} + \cdots\right).$$

By identifying the coefficients of $e^{0s}$ and $e^s$, we obtain

$$p_X(0) = \frac{2}{9}, \qquad p_X(1) = \frac{2}{27}.$$

Let $A = \{X \neq 0\}$. We have

$$p_{X|\{X\in A\}}(k) = \begin{cases} \dfrac{p_X(k)}{\mathbf{P}(A)}, & \text{if } k \neq 0, \\ 0, & \text{otherwise,} \end{cases}$$

so that

$$\begin{aligned} \mathbf{E}[X \mid X \neq 0] &= \sum_{k=1}^{\infty} k p_{X|A}(k) \\ &= \sum_{k=1}^{\infty} \frac{k p_X(k)}{\mathbf{P}(A)} \\ &= \frac{\mathbf{E}[X]}{1 - p_X(0)} \\ &= \frac{37/18}{7/9} \\ &= \frac{37}{14}. \end{aligned}$$

**Solution to Problem 4.36.** (a) We have $U = X$ if $X = 1$, which happens with probability $1/3$, and $U = Z$ if $X = 0$, which happens with probability $2/3$. Therefore, $U$ is a mixture of random variables and the associated transform is

$$M_U(s) = \mathbf{P}(X=1)M_Y(s) + \mathbf{P}(X=0)M_Z(s) = \frac{1}{3} \cdot \frac{2}{2-s} + \frac{2}{3}e^{3(e^s-1)}.$$

(b) Let $V = 2Z + 3$. We have

$$M_V(s) = e^{3s} M_Z(2s) = e^{3s} e^{3(e^{2s}-1)} = e^{3(s-1+e^{2s})}.$$

(c) Let $W = Y + Z$. We have

$$M_W(s) = M_Y(s) M_Z(s) = \frac{2}{2-s} e^{3(e^s-1)}.$$

**Solution to Problem 4.37.** Let $X$ be the number of different types of pizza ordered. Let $X_i$ be the random variable defined by

$$X_i = \begin{cases} 1, & \text{if a type } i \text{ pizza is ordered by at least one customer,} \\ 0, & \text{otherwise.} \end{cases}$$

We have $X = X_1 + \cdots + X_n$, and by the law of iterated expectations,

$$\mathbf{E}[X] = \mathbf{E}\big[\mathbf{E}[X \mid K]\big] = \mathbf{E}\big[\mathbf{E}[X_1 + \cdots + X_n \mid K]\big] = n\,\mathbf{E}\big[\mathbf{E}[X_1 \mid K]\big].$$

Furthermore, since the probability that a customer does not order a pizza of type 1 is $(n-1)/n$, we have

$$\mathbf{E}[X_1 \mid K = k] = 1 - \left(\frac{n-1}{n}\right)^k,$$

so that

$$\mathbf{E}[X_1 \mid K] = 1 - \left(\frac{n-1}{n}\right)^K.$$

Thus, denoting

$$p = \frac{n-1}{n},$$

we have

$$\mathbf{E}[X] = n\,\mathbf{E}\big[1 - p^K\big] = n - n\,\mathbf{E}\big[p^K\big] = n - n\,\mathbf{E}\big[e^{K \log p}\big] = n - nM_K(\log p).$$

**Solution to Problem 4.41.** (a) Let $N$ be the number of people that enter the elevator. The corresponding transform is $M_N(s) = e^{\lambda(e^s-1)}$. Let $M_X(s)$ be the common transform associated with the random variables $X_i$. Since $X_i$ is uniformly distributed within $[0, 1]$, we have

$$M_X(s) = \frac{e^s - 1}{s}.$$

The transform $M_Y(s)$ is found by starting with the transform $M_N(s)$ and replacing each occurrence of $e^s$ with $M_X(s)$. Thus,

$$M_Y(s) = e^{\lambda(M_X(s)-1)} = e^{\lambda\left(\frac{e^s-1}{s}-1\right)}.$$

(b) We have using the chain rule

$$\mathbf{E}[Y] = \frac{d}{ds}M_Y(s)\bigg|_{s=0} = \frac{d}{ds}M_X(s)\bigg|_{s=0} \cdot \lambda e^{\lambda(M_X(s)-1)}\bigg|_{s=0} = \frac{1}{2}\cdot\lambda = \frac{\lambda}{2},$$

where we have used the fact that $M_X(0) = 1$.

(c) From the law of iterated expectations we obtain

$$\mathbf{E}[Y] = \mathbf{E}\big[\mathbf{E}[Y \mid N]\big] = \mathbf{E}\big[N\mathbf{E}[X]\big] = \mathbf{E}[N]\mathbf{E}[X] = \frac{\lambda}{2}.$$

**Solution to Problem 4.42.** Take $X$ and $Y$ to be normal with means 1 and 2, respectively, and very small variances. Consider the random variable that takes the value of $X$ with some probability $p$ and the value of $Y$ with probability $1-p$. This random variable takes values near 1 and 2 with relatively high probability, but takes values near its mean (which is $3-2p$) with relatively low probability. Thus, this random variable is not normal.

Now let $N$ be a random variable taking only the values 1 and 2 with probabilities $p$ and $1-p$, respectively. The sum of a number $N$ of independent normal random variables with mean equal to 1 and very small variance is a mixture of the type discussed above, which is not normal.

**Solution to Problem 4.43.** (a) Using the total probability theorem, we have

$$\mathbf{P}(X > 4) = \sum_{k=0}^{4} \mathbf{P}(k \text{ lights are red})\mathbf{P}(X > 4 \mid k \text{ lights are red}).$$

We have

$$\mathbf{P}(k \text{ lights are red}) = \binom{4}{k}\left(\frac{1}{2}\right)^4.$$

The conditional PDF of $X$ given that $k$ lights are red, is normal with mean $k$ minutes and standard deviation $(1/2)\sqrt{k}$. Thus, $X$ is a mixture of normal random variables and the transform associated with its (unconditional) PDF is the corresponding mixture of the transforms associated with the (conditional) normal PDFs. However, $X$ is not normal, because a mixture of normal PDFs need not be normal. The probability $\mathbf{P}(X > 4 \mid k \text{ lights are red})$ can be computed from the normal tables for each $k$, and $\mathbf{P}(X > 4)$ is obtained by substituting the results in the total probability formula above.

(b) Let $K$ be the number of traffic lights that are found to be red. We can view $X$ as the sum of $K$ independent normal random variables. Thus the transform associated with $X$ can be found by replacing in the binomial transform $M_K(s) = (1/2 + (1/2)e^s)^4$ the occurrence of $e^s$ by the normal transform corresponding to $\mu = 1$ and $\sigma = 1/2$. Thus

$$M_X(s) = \left(\frac{1}{2} + \frac{1}{2}\left(e^{\frac{(1/2)^2 s^2}{2} + s}\right)\right)^4.$$

Note that by using the formula for the transform, we cannot easily obtain the probability $\mathbf{P}(X > 4)$.

**Solution to Problem 4.44.** (a) Using the random sum formulas, we have

$$\mathbf{E}[N] = \mathbf{E}[M]\,\mathbf{E}[K],$$

$$\text{var}(N) = \mathbf{E}[M]\,\text{var}(K) + \big(\mathbf{E}[K]\big)^2 \text{var}(M).$$

(b) Using the random sum formulas and the results of part (a), we have

$$\mathbf{E}[Y] = \mathbf{E}[N]\,\mathbf{E}[X] = \mathbf{E}[M]\,\mathbf{E}[K]\,\mathbf{E}[X],$$

$$\begin{aligned}\text{var}(Y) &= \mathbf{E}[N]\,\text{var}(X) + \big(\mathbf{E}[X]\big)^2 \text{var}(N) \\ &= \mathbf{E}[M]\,\mathbf{E}[K]\,\text{var}(X) + \big(\mathbf{E}[X]\big)^2 \Big(\mathbf{E}[M]\,\text{var}(K) + \big(\mathbf{E}[K]\big)^2 \text{var}(M)\Big).\end{aligned}$$

(c) Let $N$ denote the total number of widgets in the crate, and let $X_i$ denote the weight of the $i$th widget. The total weight of the crate is

$$Y = X_1 + \cdots + X_N,$$

with

$$N = K_1 + \cdots + K_M,$$

so the framework of part (b) applies. We have

$$\mathbf{E}[M] = \frac{1}{p}, \qquad \text{var}(M) = \frac{1-p}{p^2}, \qquad \text{(geometric formulas)},$$

$$\mathbf{E}[K] = \mu, \qquad \text{var}(M) = \mu, \qquad \text{(Poisson formulas)},$$

$$\mathbf{E}[X] = \frac{1}{\lambda}, \qquad \text{var}(M) = \frac{1}{\lambda^2}, \qquad \text{(exponential formulas)}.$$

Using these expressions into the formulas of part (b), we obtain $\mathbf{E}[Y]$ and $\text{var}(Y)$, the mean and variance of the total weight of a crate.