CHAPTER 6

Solution to Problem 6.1. (a) The random variable R is binomial with parameters p and n. Hence,

$$p_R(r) = \binom{n}{r}(1-p)^{n-r}p^r, \qquad \text{for } r = 0, 1, 2, \ldots, n,$$

$\mathbf{E}[R] = np$, and $\text{var}(R) = np(1-p)$.

(b) Let A be the event that the first item to be loaded ends up being the only one on its truck. This event is the union of two disjoint events:

(i) the first item is placed on the red truck and the remaining $n-1$ are placed on the green truck, and,

(ii) the first item is placed on the green truck and the remaining $n-1$ are placed on the red truck.

Thus, $\mathbf{P}(A) = p(1-p)^{n-1} + (1-p)p^{n-1}$.

(c) Let B be the event that at least one truck ends up with a total of exactly one package. The event B occurs if exactly one or both of the trucks end up with exactly 1 package, so

$$\mathbf{P}(B) = \begin{cases} 1, & \text{if } n = 1, \\ 2p(1-p), & \text{if } n = 2, \\ \binom{n}{1}(1-p)^{n-1}p + \binom{n}{n-1}p^{n-1}(1-p), & \text{if } n = 3, 4, 5, \ldots \end{cases}$$

(d) Let $D = R - G = R - (n - R) = 2R - n$. We have $\mathbf{E}[D] = 2\mathbf{E}[R] - n = 2np - n$. Since $D = 2R - n$, where n is a constant,

$$\text{var}(D) = 4\text{var}(R) = 4np(1-p).$$

(e) Let C be the event that each of the first 2 packages is loaded onto the red truck. Given that C occurred, the random variable R becomes

$$2 + X_3 + X_4 + \cdots + X_n.$$

Hence,

$$\mathbf{E}[R \,|\, C] = \mathbf{E}[2 + X_3 + X_4 + \cdots + X_n] = 2 + (n-2)\mathbf{E}[X_i] = 2 + (n-2)p.$$

Similarly, the conditional variance of R is

$$\text{var}(R \,|\, C) = \text{var}(2 + X_3 + X_4 + \cdots + X_n) = (n-2)\text{var}(X_i) = (n-2)p(1-p).$$

Finally, given that the first two packages are loaded onto the red truck, the probability that a total of r packages are loaded onto the red truck is equal to the probability that $r-2$ of the remaining $n-2$ packages go to the red truck:

$$p_{R|C}(r) = \binom{n-2}{r-2}(1-p)^{n-r}p^{r-2}, \qquad \text{for } r = 2, \ldots, n.$$

Solution to Problem 6.2. (a) Failed quizzes are a Bernoulli process with parameter $p = 1/4$. The desired probability is given by the binomial formula:

$$\binom{6}{2}p^2(1-p)^4 = \frac{6!}{4!\,2!}\left(\frac{1}{4}\right)^2\left(\frac{3}{4}\right)^4.$$

(b) The expected number of quizzes up to the third failure is the expected value of a Pascal random variable of order three, with parameter 1/4, which is $3 \cdot 4 = 12$. Subtracting the number of failures, we have that the expected number of quizzes that Dave will pass is $12 - 3 = 9$.

(c) The event of interest is the intersection of the following three independent events:

A: there is exactly one failure in the first seven quizzes.

B: quiz eight is a failure.

C: quiz nine is a failure.

We have

$$\mathbf{P}(A) = \binom{7}{1}\left(\frac{1}{4}\right)\left(\frac{3}{4}\right)^6, \qquad \mathbf{P}(B) = \mathbf{P}(C) = \frac{1}{4},$$

so the desired probability is

$$\mathbf{P}(A \cap B \cap C) = 7\left(\frac{1}{4}\right)^3\left(\frac{3}{4}\right)^6.$$

(d) Let B be the event that Dave fails two quizzes in a row before he passes two quizzes in a row. Let us use F and S to indicate quizzes that he has failed or passed, respectively. We then have

$$\begin{aligned}
\mathbf{P}(B) &= \mathbf{P}(FF \cup SFF \cup FSFF \cup SFSFF \cup FSFSFF \cup SFSFSFF \cup \cdots) \\
&= \mathbf{P}(FF) + \mathbf{P}(SFF) + \mathbf{P}(FSFF) + \mathbf{P}(SFSFF) + \mathbf{P}(FSFSFF) \\
&\quad + \mathbf{P}(SFSFSFF) + \cdots \\
&= \left(\frac{1}{4}\right)^2 + \frac{3}{4}\left(\frac{1}{4}\right)^2 + \frac{1}{4}\cdot\frac{3}{4}\left(\frac{1}{4}\right)^2 + \frac{3}{4}\cdot\frac{1}{4}\cdot\frac{3}{4}\left(\frac{1}{4}\right)^2 + \frac{1}{4}\cdot\frac{3}{4}\cdot\frac{1}{4}\cdot\frac{3}{4}\left(\frac{1}{4}\right)^2 \\
&\quad + \frac{3}{4}\cdot\frac{1}{4}\cdot\frac{3}{4}\cdot\frac{1}{4}\cdot\frac{3}{4}\left(\frac{1}{4}\right)^2 + \cdots \\
&= \left[\left(\frac{1}{4}\right)^2 + \frac{1}{4}\cdot\frac{3}{4}\left(\frac{1}{4}\right)^2 + \frac{1}{4}\cdot\frac{3}{4}\cdot\frac{1}{4}\cdot\frac{3}{4}\left(\frac{1}{4}\right)^2 + \cdots\right] \\
&\quad + \left[\frac{3}{4}\left(\frac{1}{4}\right)^2 + \frac{3}{4}\cdot\frac{1}{4}\cdot\frac{3}{4}\left(\frac{1}{4}\right)^2 + \frac{3}{4}\cdot\frac{1}{4}\cdot\frac{3}{4}\cdot\frac{1}{4}\cdot\frac{3}{4}\left(\frac{1}{4}\right)^2 + \cdots\right].
\end{aligned}$$

Therefore, $\mathbf{P}(B)$ is the sum of two infinite geometric series, and

$$\mathbf{P}(B) = \frac{\left(\frac{1}{4}\right)^2}{1 - \frac{1}{4} \cdot \frac{3}{4}} + \frac{\frac{3}{4}\left(\frac{1}{4}\right)^2}{1 - \frac{3}{4} \cdot \frac{1}{4}} = \frac{7}{52}.$$

Solution to Problem 6.3. The answers to these questions are found by considering suitable Bernoulli processes and using the formulas of Section 6.1. Depending on the specific question, however, a different Bernoulli process may be appropriate. In some cases, we associate trials with slots. In other cases, it is convenient to associate trials with busy slots.

(a) During each slot, the probability of a task from user 1 is given by $p_1 = p_{1|B} p_B = (5/6) \cdot (2/5) = 1/3$. Tasks from user 1 form a Bernoulli process and

$$\mathbf{P}(\text{first user 1 task occurs in slot 4}) = p_1(1 - p_1)^3 = \frac{1}{3} \cdot \left(\frac{2}{3}\right)^3.$$

(b) This is the probability that slot 11 was busy and slot 12 was idle, given that 5 out of the 10 first slots were idle. Because of the fresh-start property, the conditioning information is immaterial, and the desired probability is

$$p_B \cdot p_I = \frac{5}{6} \cdot \frac{1}{6}.$$

(c) Each slot contains a task from user 1 with probability $p_1 = 1/3$, independent of other slots. The time of the 5th task from user 1 is a Pascal random variable of order 5, with parameter $p_1 = 1/3$. Its mean is given by

$$\frac{5}{p_1} = \frac{5}{1/3} = 15.$$

(d) Each busy slot contains a task from user 1 with probability $p_{1|B} = 2/5$, independent of other slots. The random variable of interest is a Pascal random variable of order 5, with parameter $p_{1|B} = 2/5$. Its mean is

$$\frac{5}{p_{1|B}} = \frac{5}{2/5} = \frac{25}{2}.$$

(e) The number T of tasks from user 2 until the 5th task from user 1 is the same as the number B of busy slots until the 5th task from user 1, minus 5. The number of busy slots ("trials") until the 5th task from user 1 ("success") is a Pascal random variable of order 5, with parameter $p_{1|B} = 2/5$. Thus,

$$p_B(t) = \binom{t-1}{4} \left(\frac{2}{5}\right)^5 \left(1 - \frac{2}{5}\right)^{t-5}, \qquad t = 5, 6, \ldots.$$

Since $T = B - 5$, we have $p_T(t) = p_B(t+5)$, and we obtain

$$p_T(t) = \binom{t+4}{4} \left(\frac{2}{5}\right)^5 \left(1 - \frac{2}{5}\right)^t, \qquad t = 0, 1, \ldots.$$

Using the formulas for the mean and the variance of the Pascal random variable B, we obtain

$$\mathbf{E}[T] = \mathbf{E}[B] - 5 = \frac{25}{2} - 5 = 7.5,$$

and

$$\text{var}(T) = \text{var}(B) = \frac{5\big(1 - (2/5)\big)}{(2/5)^2}.$$

Solution to Problem 6.8. The total number of accidents between 8 am and 11 am is the sum of two independent Poisson random variables with parameters 5 and $3 \cdot 2 = 6$, respectively. Since the sum of independent Poisson random variables is also Poisson, the total number of accidents has a Poisson PMF with parameter 5+6=11.

Solution to Problem 6.9. As long as the pair of players is waiting, all five courts are occupied by other players. When all five courts are occupied, the time until a court is freed up is exponentially distributed with mean $40/5 = 8$ minutes. For our pair of players to get a court, a court must be freed up $k+1$ times. Thus, the expected waiting time is $8(k+1)$.

Solution to Problem 6.10. (a) This is the probability of no arrivals in 2 hours. It is given by

$$P(0,2) = e^{-0.6 \cdot 2} = 0.301.$$

For an alternative solution, this is the probability that the first arrival comes after 2 hours:

$$\mathbf{P}(T_1 > 2) = \int_2^\infty f_{T_1}(t)\,dt = \int_2^\infty 0.6e^{-0.6t}\,dt = e^{-0.6 \cdot 2} = 0.301.$$

(b) This is the probability of zero arrivals between time 0 and 2, and of at least one arrival between time 2 and 5. Since these two intervals are disjoint, the desired probability is the product of the probabilities of these two events, which is given by

$$P(0,2)\big(1 - P(0,3)\big) = e^{-0.6 \cdot 2}(1 - e^{-0.6 \cdot 3}) = 0.251.$$

For an alternative solution, the event of interest can be written as $\{2 \le T_1 \le 5\}$, and its probability is

$$\int_2^5 f_{T_1}(t)\,dt = \int_2^5 0.6e^{-0.6t}\,dt = e^{-0.6 \cdot 2} - e^{-0.6 \cdot 5} = 0.251.$$

(c) If he catches at least two fish, he must have fished for exactly two hours. Hence, the desired probability is equal to the probability that the number of fish caught in the first two hours is at least two, i.e.,

$$\sum_{k=2}^\infty P(k,2) = 1 - P(0,2) - P(1,2) = 1 - e^{-0.6 \cdot 2} - (0.6 \cdot 2)e^{-0.6 \cdot 2} = 0.337.$$

For an alternative approach, note that the event of interest occurs if and only if the time Y_2 of the second arrival is less than or equal to 2. Hence, the desired probability is

$$\mathbf{P}(Y_2 \le 2) = \int_0^2 f_{Y_2}(y)\,dy = \int_0^2 (0.6)^2 y e^{-0.6y}\,dy.$$

This integral can be evaluated by integrating by parts, but this is more tedious than the first approach.

(d) The expected number of fish caught is equal to the expected number of fish caught during the first two hours (which is $2\lambda = 2 \cdot 0.6 = 1.2$), plus the expectation of the number N of fish caught after the first two hours. We have $N = 0$ if he stops fishing at two hours, and $N = 1$, if he continues beyond the two hours. The event $\{N = 1\}$ occurs if and only if no fish are caught in the first two hours, so that $\mathbf{E}[N] = \mathbf{P}(N = 1) = P(0, 2) = 0.301$. Thus, the expected number of fish caught is $1.2 + 0.301 = 1.501$.

(e) Given that he has been fishing for 4 hours, the future fishing time is the time until the first fish is caught. By the memorylessness property of the Poisson process, the future time is exponential, with mean $1/\lambda$. Hence, the expected total fishing time is $4 + (1/0.6) = 5.667$.

Solution to Problem 6.11. We note that the process of departures of customers who have bought a book is obtained by splitting the Poisson process of customer departures, and is itself a Poisson process, with rate $p\lambda$.

(a) This is the time until the first customer departure in the split Poisson process. It is therefore exponentially distributed with parameter $p\lambda$.

(b) This is the probability of no customers in the split Poisson process during an hour, and using the result of part (a), equals $e^{-p\lambda}$.

(c) This is the expected number of customers in the split Poisson process during an hour, and is equal to $p\lambda$.

Solution to Problem 6.12. Let X be the number of different types of pizza ordered. Let X_i be the random variable defined by

$$X_i = \begin{cases} 1, & \text{if a type } i \text{ pizza is ordered by at least one customer,} \\ 0, & \text{otherwise.} \end{cases}$$

We have $X = X_1 + \cdots + X_n$, and $\mathbf{E}[X] = n\mathbf{E}[X_1]$.

We can think of the customers arriving as a Poisson process, and with each customer independently choosing whether to order a type 1 pizza (this happens with probability $1/n$) or not. This is the situation encountered in splitting of Poisson processes, and the number of type 1 pizza orders, denoted Y_1, is a Poisson random variable with parameter λ/n. We have

$$\mathbf{E}[X_1] = \mathbf{P}(Y_1 > 0) = 1 - \mathbf{P}(Y_1 = 0) = 1 - e^{-\lambda/n},$$

so that

$$\mathbf{E}[X] = n\mathbf{E}[X_1] = n\big(1 - e^{-\lambda/n}\big).$$

Solution to Problem 6.13. (a) Let R be the total number of messages received during an interval of duration t. Note that R is a Poisson random variable with arrival rate $\lambda_A + \lambda_B$. Therefore, the probability that exactly nine messages are received is

$$\mathbf{P}(R = 9) = \frac{\big((\lambda_A + \lambda_B)T\big)^9 e^{-(\lambda_A+\lambda_B)t}}{9!}.$$

(b) Let R be defined as in part (a), and let W_i be the number of words in the ith message. Then,

$$N = W_1 + W_2 + \cdots + W_R,$$

which is a sum of a random number of random variables. Thus,

$$\begin{aligned}\mathbf{E}[N] &= \mathbf{E}[W]\mathbf{E}[R] \\ &= \left(1 \cdot \frac{2}{6} + 2 \cdot \frac{3}{6} + 3 \cdot \frac{1}{6}\right)(\lambda_A + \lambda_B)t \\ &= \frac{11}{6}(\lambda_A + \lambda_B)t.\end{aligned}$$

(c) Three-word messages arrive from transmitter A in a Poisson manner, with average rate $\lambda_A p_W(3) = \lambda_A/6$. Therefore, the random variable of interest is Erlang of order 8, and its PDF is given by

$$f(x) = \frac{(\lambda_A/6)^8 x^7 e^{-\lambda_A x/6}}{7!}.$$

(d) Every message originates from either transmitter A or B, and can be viewed as an independent Bernoulli trial. Each message has probability $\lambda_A/(\lambda_A+\lambda_B)$ of originating from transmitter A (view this as a "success"). Thus, the number of messages from transmitter A (out of the next twelve) is a binomial random variable, and the desired probability is equal to

$$\binom{12}{8}\left(\frac{\lambda_A}{\lambda_A + \lambda_B}\right)^8\left(\frac{\lambda_B}{\lambda_A + \lambda_B}\right)^4.$$

Solution to Problem 6.14. (a) Let X be the time until the first bulb failure. Let A (respectively, B) be the event that the first bulb is of type A (respectively, B). Since the two bulb types are equally likely, the total expectation theorem yields

$$\mathbf{E}[X] = \mathbf{E}[X \mid A]\mathbf{P}(A) + \mathbf{E}[X \mid B]\mathbf{P}(B) = 1 \cdot \frac{1}{2} + \frac{1}{3} \cdot \frac{1}{2} = \frac{2}{3}.$$

(b) Let D be the event of no bulb failures before time t. Using the total probability theorem, and the exponential distributions for bulbs of the two types, we obtain

$$\mathbf{P}(D) = \mathbf{P}(D \mid A)\mathbf{P}(A) + \mathbf{P}(D \mid B)\mathbf{P}(B) = \frac{1}{2}e^{-t} + \frac{1}{2}e^{-3t}.$$

(c) We have

$$\mathbf{P}(A \mid D) = \frac{\mathbf{P}(A \cap D)}{\mathbf{P}(D)} = \frac{\frac{1}{2}e^{-t}}{\frac{1}{2}e^{-t} + \frac{1}{2}e^{-3t}} = \frac{1}{1 + e^{-2t}}.$$

(d) We first find $\mathbf{E}[X^2]$. We use the fact that the second moment of an exponential random variable T with parameter λ is equal to $\mathbf{E}[T^2] = \mathbf{E}[T]^2 + \text{var}(T) = 1/\lambda^2 + 1/\lambda^2 = 2/\lambda^2$. Conditioning on the two possible types of the first bulb, we obtain

$$\mathbf{E}[X^2] = \mathbf{E}[X^2 \mid A]\mathbf{P}(A) + \mathbf{E}[X^2 \mid B]\mathbf{P}(B) = 2 \cdot \frac{1}{2} + \frac{2}{9} \cdot \frac{1}{2} = \frac{10}{9}.$$

Finally, using the fact $\mathbf{E}[X] = 2/3$ from part (a),

$$\text{var}(X) = \mathbf{E}[X^2] - \mathbf{E}[X]^2 = \frac{10}{9} - \frac{2^2}{3^2} = \frac{2}{3}.$$

(e) This is the probability that out of the first 11 bulbs, exactly 3 were of type A and that the 12th bulb was of type A. It is equal to

$$\binom{11}{3}\left(\frac{1}{2}\right)^{12}.$$

(f) This is the probability that out of the first 12 bulbs, exactly 4 were of type A, and is equal to

$$\binom{12}{4}\left(\frac{1}{2}\right)^{12}.$$

(g) The PDF of the time between failures is $(e^{-x} + 3e^{-3x})/2$, for $x \geq 0$, and the associated transform is

$$\frac{1}{2}\left(\frac{1}{1-s} + \frac{3}{3-s}\right).$$

Since the times between successive failures are independent, the transform associated with the time until the 12th failure is given by

$$\left[\frac{1}{2}\left(\frac{1}{1-s} + \frac{3}{3-s}\right)\right]^{12}.$$

(h) Let Y be the total period of illumination provided by the first two type-B bulbs. This has an Erlang distribution of order 2, and its PDF is

$$f_Y(y) = 9ye^{-3y}, \qquad y \geq 0.$$

Let T be the period of illumination provided by the first type-A bulb. Its PDF is

$$f_T(t) = e^{-t}, \qquad t \geq 0.$$

We are interested in the event $T < Y$. We have

$$\mathbf{P}(T < Y \,|\, Y = y) = 1 - e^{-y}, \qquad y \geq 0.$$

Thus,

$$\mathbf{P}(T < Y) = \int_0^\infty f_Y(y)\mathbf{P}(T < Y \,|\, Y = y)\, dy = \int_0^\infty 9ye^{-3y}\big(1 - e^{-y}\big)\, dy = \frac{7}{16},$$

as can be verified by carrying out the integration.

We now describe an alternative method for obtaining the answer. Let T_1^A be the period of illumination of the first type-A bulb. Let T_1^B and T_2^B be the period of illumination provided by the first and second type-B bulb, respectively. We are interested in the event $\{T_1^A < T_1^B + T_2^B\}$. We have

$$\begin{aligned}
\mathbf{P}(T_1^A < T_1^B + T_2^B) &= \mathbf{P}(T_1^A < T_1^B) + \mathbf{P}(T_1^A \geq T_1^B)\,\mathbf{P}(T_1^A < T_1^B + T_2^B \,|\, T_1^A \geq T_1^B) \\
&= \frac{1}{1+3} + \mathbf{P}(T_1^A \geq T_1^B)\,\mathbf{P}(T_1^A - T_1^B < T_2^B \,|\, T_1^A \geq T_1^B) \\
&= \frac{1}{4} + \frac{3}{4}\mathbf{P}(T_1^A - T_1^B < T_2^B \,|\, T_1^A \geq T_1^B).
\end{aligned}$$

Given the event $T_1^A \geq T_1^B$, and using the memorylessness property of the exponential random variable T_1^A, the remaining time $T_1^A - T_1^B$ until the failure of the type-A bulb is exponentially distributed, so that

$$\mathbf{P}(T_1^A - T_1^B < T_2^B \,|\, T_1^A \geq T_1^B) = \mathbf{P}(T_1^A < T_2^B) = \mathbf{P}(T_1^A < T_1^B) = \frac{1}{4}.$$

Therefore,

$$\mathbf{P}(T_1^A < T_1^B + T_2^B) = \frac{1}{4} + \frac{3}{4} \cdot \frac{1}{4} = \frac{7}{16}.$$

(i) Let V be the total period of illumination provided by type-B bulbs while the process is in operation. Let N be the number of light bulbs, out of the first 12, that are of type B. Let X_i be the period of illumination from the ith type-B bulb. We then have $V = Y_1 + \cdots + Y_N$. Note that N is a binomial random variable, with parameters $n = 12$ and $p = 1/2$, so that

$$\mathbf{E}[N] = 6, \qquad \text{var}(N) = 12 \cdot \frac{1}{2} \cdot \frac{1}{2} = 3.$$

Furthermore, $\mathbf{E}[X_i] = 1/3$ and $\text{var}(X_i) = 1/9$. Using the formulas for the mean and variance of the sum of a random number of random variables, we obtain

$$\mathbf{E}[V] = \mathbf{E}[N]\mathbf{E}[X_i] = 2,$$

and

$$\text{var}(V) = \text{var}(X_i)\mathbf{E}[N] + \mathbf{E}[X_i]^2\text{var}(N) = \frac{1}{9} \cdot 6 + \frac{1}{9} \cdot 3 = 1.$$

(j) Using the notation in parts (a)-(c), and the result of part (c), we have

$$\begin{aligned}\mathbf{E}[T \mid D] &= t + \mathbf{E}[T - t \mid D \cap A]\mathbf{P}(A \mid D) + \mathbf{E}[T - t \mid D \cap B]\mathbf{P}(B \mid D) \\ &= t + 1 \cdot \frac{1}{1 + e^{-2t}} + \frac{1}{3}\left(1 - \frac{1}{1 + e^{-2t}}\right) \\ &= t + \frac{1}{3} + \frac{2}{3} \cdot \frac{1}{1 + e^{-2t}}.\end{aligned}$$

Solution to Problem 6.15. (a) The total arrival process corresponds to the merging of two independent Poisson processes, and is therefore Poisson with rate $\lambda = \lambda_A + \lambda_B = 7$. Thus, the number N of jobs that arrive in a given three-minute interval is a Poisson random variable, with $\mathbf{E}[N] = 3\lambda = 21$, $\text{var}(N) = 21$, and PMF

$$p_N(n) = \frac{(21)^n e^{-21}}{n!}, \qquad n = 0, 1, 2, \ldots.$$

(b) Each of these 10 jobs has probability $\lambda_A/(\lambda_A + \lambda_B) = 3/7$ of being of type A, independently of the others. Thus, the binomial PMF applies and the desired probability is equal to

$$\binom{10}{3} \left(\frac{3}{7}\right)^3 \left(\frac{4}{7}\right)^7.$$

(c) Each future arrival is of type A with probability $\lambda_A/(\lambda_A + \lambda_B) = 3/7$, independently of other arrivals. Thus, the number K of arrivals until the first type A arrival is geometric with parameter 3/7. The number of type B arrivals before the first type A arrival is equal to $K - 1$, and its PMF is similar to a geometric, except that it is shifted by one unit to the left. In particular,

$$p_K(k) = \left(\frac{3}{7}\right) \left(\frac{4}{7}\right)^k, \qquad k = 0, 1, 2, \ldots.$$

(d) The fact that at time 0 there were two type A jobs in the system simply states that there were exactly two type A arrivals between time -1 and time 0. Let X and Y be the arrival times of these two jobs. Consider splitting the interval $[-1, 0]$ into many time slots of length δ. Since each time instant is equally likely to contain an arrival and since the arrival times are independent, it follows that X and Y are independent uniform random variables. We are interested in the PDF of $Z = \max\{X, Y\}$. We first find the CDF of Z. We have, for $z \in [-1, 0]$,

$$\mathbf{P}(Z \leq z) = \mathbf{P}(X \leq z \text{ and } Y \leq z) = (1 + z)^2.$$

By differentiating, we obtain

$$f_Z(z) = 2(1 + z), \qquad -1 \leq z \leq 0.$$

(e) Let T be the arrival time of this type B job. We can express T in the form $T = -K + X$, where K is a nonnegative integer and X lies in [0,1]. We claim that X

is independent from K and that X is uniformly distributed. Indeed, conditioned on the event $K = k$, we know that there was a single arrival in the interval $[-k, -k+1]$. Conditioned on the latter information, the arrival time is uniformly distributed in the interval $[-k, k+1]$ (cf. Problem 6.18), which implies that X isuniformly distributed in $[0, 1]$. Since this conditional distribution of X is the same for every k, it follows that X is independent of $-K$.

Let D be the departure time of the job of interest. Since the job stays in the system for an integer amount of time, we have that D is of the form $D = L + X$, where L is a nonnegative integer. Since the job stays in the system for a geometrically distributed amount of time, and the geometric distribution has the memorylessness property, it follows that L is also memoryless. In particular, L is similar to a geometric random variable, except that its PMF starts at zero. Furthermore, L is independent of X, since X is determined by the arrival process, whereas the amount of time a job stays in the system is independent of the arrival process. Thus, D is the sum of two independent random variables, one uniform and one geometric. Therefore, D has "geometric staircase" PDF, given by

$$f_D(d) = \left(\frac{1}{2}\right)^{\lfloor d \rfloor}, \qquad d \geq 0,$$

and where $\lfloor d \rfloor$ stands for the largest integer below d.

Solution to Problem 6.16. (a) The random variable N is equal to the number of successive interarrival intervals that are smaller than τ. Interarrival intervals are independent and each one is smaller than τ with probability $1 - e^{-\lambda\tau}$. Therefore,

$$\mathbf{P}(N = 0) = e^{-\lambda\tau}, \qquad \mathbf{P}(N = 1) = e^{-\lambda\tau}\left(1 - e^{-\lambda\tau}\right), \qquad \mathbf{P}(N = k) = e^{-\lambda\tau}\left(1 - e^{-\lambda\tau}\right)^k,$$

so that N has a distribution similar to a geometric one, with parameter $p = e^{-\lambda\tau}$, except that it shifted one place to the left, so that it starts out at 0. Hence,

$$\mathbf{E}[N] = \frac{1}{p} - 1 = e^{\lambda\tau} - 1.$$

(b) Let T_n be the nth interarrival time. The event $\{N \geq n\}$ indicates that the time between cars $n-1$ and n is less than or equal to τ, and therefore $\mathbf{E}[T_n \,|\, N \geq n] = \mathbf{E}[T_n \,|\, T_n \leq \tau]$. Note that the conditional PDF of T_n is the same as the unconditional one, except that it is now restricted to the interval $[0, \tau]$, and that it has to be suitably renormalized so that it integrates to 1. Therefore, the desired conditional expectation is

$$\mathbf{E}[T_n \,|\, T_n \leq \tau] = \frac{\displaystyle\int_0^\tau s\lambda e^{-\lambda s}\, ds}{\displaystyle\int_0^\tau \lambda e^{-\lambda s}\, ds}.$$

This integral can be evaluated by parts. We will provide, however, an alternative approach that avoids integration.

We use the total expectation formula

$$\mathbf{E}[T_n] = \mathbf{E}[T_n \,|\, T_n \leq \tau]\mathbf{P}(T_n \leq \tau) + \mathbf{E}[T_n \,|\, T_n > \tau]\mathbf{P}(T_n > \tau).$$

We have $\mathbf{E}[T_n] = 1/\lambda$, $\mathbf{P}(T_n \leq \tau) = 1 - e^{-\lambda\tau}$, $\mathbf{P}(T_n > \tau) = e^{-\lambda\tau}$, and $\mathbf{E}[T_n \,|\, T_n > \tau] = \tau + (1/\lambda)$. (The last equality follows from the memorylessness of the exponential PDF.) Using these equalities, we obtain

$$\frac{1}{\lambda} = \mathbf{E}[T_n \,|\, T_n \leq \tau]\big(1 - e^{-\lambda\tau}\big) + \left(\tau + \frac{1}{\lambda}\right) e^{-\lambda\tau},$$

which yields

$$\mathbf{E}[T_n \,|\, T_n \leq \tau] = \frac{\dfrac{1}{\lambda} - \left(\tau + \dfrac{1}{\lambda}\right) e^{-\lambda\tau}}{1 - e^{-\lambda\tau}}.$$

(c) Let T be the time until the U-turn. Note that $T = T_1 + \cdots + T_N + \tau$. Let v denote the value of $\mathbf{E}[T_n \,|\, T_n \leq \tau]$. We find $\mathbf{E}[T]$ using the total expectation theorem:

$$\begin{aligned}
\mathbf{E}[T] &= \tau + \sum_{n=0}^{\infty} \mathbf{P}(N = n)\mathbf{E}[T_1 + \cdots + T_N \,|\, N = n] \\
&= \tau + \sum_{n=0}^{\infty} \mathbf{P}(N = n) \sum_{i=1}^{n} \mathbf{E}[T_i \,|\, T_1 \leq \tau, \ldots, T_n \leq \tau, T_{n+1} > \tau] \\
&= \tau + \sum_{n=0}^{\infty} \mathbf{P}(N = n) \sum_{i=1}^{n} \mathbf{E}[T_i \,|\, T_i \leq \tau] \\
&= \tau + \sum_{n=0}^{\infty} \mathbf{P}(N = n) n v \\
&= \tau + v\mathbf{E}[N],
\end{aligned}$$

where $\mathbf{E}[N]$ was found in part (a) and v was found in part (b). The second equality used the fact that the event $\{N = n\}$ is the same as the event $\{T_1 \leq \tau, \ldots, T_n \leq \tau, T_{n+1} > \tau\}$. The third equality used the independence of the interarrival times T_i.

Solution to Problem 6.17. We will calculate the expected length of the photographer's waiting time T conditioned on each of the two events: A, which is that the photographer arrives while the wombat is resting or eating, and A^c, which is that the photographer arrives while the wombat is walking. We will then use the total expectation theorem as follows:

$$\mathbf{E}[T] = \mathbf{P}(A)\mathbf{E}[T \,|\, A] + \mathbf{P}(A^c)\mathbf{E}[T \,|\, A^c].$$

The conditional expectation $\mathbf{E}[T \,|\, A]$ can be broken down in three components:

(i) The expected remaining time up to when the wombat starts its next walk; by the memorylessness property, this time is exponentially distributed and its expected value is 30 secs.

(ii) A number of walking and resting/eating intervals (each of expected length 50 secs) during which the wombat does not stop; if N is the number of these intervals, then $N + 1$ is geometrically distributed with parameter $1/3$. Thus the expected length of these intervals is $(3 - 1) \cdot 50 = 100$ secs.

(iii) The expected waiting time during the walking interval in which the wombat stands still. This time is uniformly distributed between 0 and 20, so its expected value is 10 secs.

Collecting the above terms, we see that

$$\mathbf{E}[T \mid A] = 30 + 100 + 10 = 140.$$

The conditional expectation $\mathbf{E}[T \mid A^c]$ can be calculated using the total expectation theorem, by conditioning on three events: B_1, which is that the wombat does not stop during the photographer's arrival interval (probability 2/3); B_2, which is that the wombat stops during the photographer's arrival interval after the photographer arrives (probability 1/6); B_3, which is that the wombat stops during the photographer's arrival interval before the photographer arrives (probability 1/6). We have

$$\begin{aligned}\mathbf{E}[T \mid A^c, B_1] &= \mathbf{E}[\text{photographer's wait up to the end of the interval}] + \mathbf{E}[T \mid A] \\ &= 10 + 140 = 150.\end{aligned}$$

Also, it can be shown that if two points are randomly chosen in an interval of length l, the expected distance between the two points is $l/3$ (an end-of-chapter problem in Chapter 3), and using this fact, we have

$$\mathbf{E}[T \mid A^c, B_2] = \mathbf{E}[\text{photographer's wait up to the time when the wombat stops}] = \frac{20}{3}.$$

Similarly, it can be shown that if two points are randomly chosen in an interval of length l, the expected distance between each point and the nearest endpoint of the interval is $l/3$. Using this fact, we have

$$\begin{aligned}\mathbf{E}[T \mid A^c, B_3] &= \mathbf{E}[\text{photographer's wait up to the end of the interval}] + \mathbf{E}[T \mid A] \\ &= \frac{20}{3} + 140.\end{aligned}$$

Applying the total expectation theorem, we see that

$$\mathbf{E}[T \mid A^c] = \frac{2}{3} \cdot 150 + \frac{1}{6} \cdot \frac{20}{3} + \frac{1}{6}\left(\frac{20}{3} + 140\right) = 125.55.$$

To apply the total expectation theorem and obtain $\mathbf{E}[T]$, we need the probability $\mathbf{P}(A)$ that the photographer arrives during a resting/eating interval. Since the expected length of such an interval is 30 seconds and the length of the complementary walking interval is 20 seconds, we see that $\mathbf{P}(A) = 30/50 = 0.6$. Substituting in the equation

$$\mathbf{E}[T] = \mathbf{P}(A)\mathbf{E}[T \mid A] + \big(1 - \mathbf{P}(A)\big)\mathbf{E}[T \mid A^c],$$

we obtain

$$\mathbf{E}[T] = 0.6 \cdot 140 + 0.4 \cdot 125.55 = 134.22.$$