Introduction to Probability Midterm 2011.4.18

1. (10%) Alvin's database of friends contains n entries. Due to a software glitch, the addresses correspond to the names in a totally random fashion. Alvin writes a holiday card to each to his friends and sends it to the address. What is the probability that at least one of his friends receives the correct card?

   HW 2.30 Sol.

   Let A_k be the event that the kth card is sent to the correct address. We have for any k, j, i,

$$P(A_k) = \frac{1}{n}$$

$$P(A_k \cap A_j) = P(A_k)P(A_j \mid A_k) = \frac{1}{n}\frac{1}{n-1} = \frac{(n-2)!}{n!}$$

$$P(A_k \cap A_j \cap A_i) = \frac{1}{n}\frac{1}{n-1}\frac{1}{n-2} = \frac{(n-3)!}{n!}$$

$$P(\bigcup_{k=1}^{n} A_k) = C_1^n\frac{(n-1)!}{n!} - C_2^n\frac{(n-2)!}{n!} + \cdots + (-1)^{n-1}\frac{1}{n!}$$

$$= 1 - \frac{1}{2!} + \frac{1}{3!} + \cdots + (-1)^n\frac{1}{n!}$$

2. (10%) A smoker mathematician carries one matchbox in the left pocket and one matchbox in his right pocket. Initially, both boxes have n matches. Each time he wants to light a cigarette, he selects with equal probability to use the left or right matchbox. What is the PMF of the number of matches in the remaining matchbox when he finds the matchbox he chooses is empty?

   HW 2.11

$$p_X(k) = P(L_k) + P(R_k)$$

$$P(L_k) = \frac{1}{2}C_n^{2n-k}(\frac{1}{2})^{2n-k}$$

$$p_X(k) = P(L_k) + P(R_k) = C_n^{2n-k}(\frac{1}{2})^{2n-k}$$

3. (10%) The MIT soccer team is scheduled to play two games in one weekend. It is a 0.4 probability of not losing in the first game, and a 0.7 probability of not losing in the second game. The team receive 2 points for a win, 1 point for a tie, and 0 for a loss. When the team does not lose in a game, it is equally likely to win or draw. Find the PMF of the number of points earned over the weekend.

   HW 2.1

$$P(x=0) = 0.6 * 0.3 * 0.18$$
$$P(x=1) = 0.4 * 0.5 * 0.3 + 0.6 * 0.5 * 0.7 = 0.27$$
$$P(x=2) = 0.4 * 0.5 * 0.3 + 0.6 * 0.5 * 0.7 + 0.4 + 0.5 * 0.7 * 0.5 = 0.34$$
$$P(x=3) = 0.4 * 0.5 * 0.7 * 0.5 + 0.4 * 0.5 * 0.7 * 0.5 = 0.14$$
$$P(x=4) = 0.4 * 0.5 * 0.7 * 0.5 = 0.07$$
$$P(x>4) = 0$$

4. (10%) A die with r faces is rolled n times. In each roll, face i is up with probability p_i. Let X_i be the number of times that face i is up in n rolls. Find

   (a) the expected value and variance of X_i, and

   (b) $E[X_i X_j]$ for $i \neq j$.

   HW 2.27 Sol.

$$E[x_i] = np_i$$
$$\begin{aligned} E[x_i y_i] &= E[(Y_{i,1} + \cdots + Y_{i,n})(Y_{j,1} + \cdots + Y_{j,n})] \\ &= n(n-1)E[Y_{i,1}Y_{j,2}] \\ &= n(n-1)p_i p_j \end{aligned}$$

5. (10%) Let X has the PDF

$$f_X(x) = \frac{\lambda}{2} e^{-\lambda|x|}, \quad \lambda > 0.$$

Verify that $f_X(x)$ is a valid PDF and find the mean and variance of X.

HW 3.2 Sol.

$$\int_{-\infty}^{\infty} f_x(x)dx = \int_{-\infty}^{\infty} \frac{\lambda}{2} e^{-\lambda|x|} = 2 * \frac{1}{2} \int_{-\infty}^{\infty} \lambda e^{-\lambda x} = 1$$

$$(\because E[x] = 0)$$

$$E[x^2] = \int_{-\infty}^{\infty} x^2 \frac{\lambda}{2} e^{-\lambda|x|} = \int_{0}^{\infty} x^2 \lambda e^{-\lambda x} = \frac{2}{\lambda^2}$$

$$Var(X) = E[x^2] - (E[x])^2 = \frac{2}{\lambda^2}$$

6. (10%) The taxi stand and the bus stop near Al's home are in the same location. Al goes there at a given time and if a taxi is waiting (with probability $2/3$) he boards it. Otherwise he waits for a taxi or a bus, whichever comes first. The next bus always arrives in 5 minutes, while the next taxi arrives between 0 and 10 minutes, uniformly distributed. Find the CDF and the expected value of Al's waiting time.

HW 3.9 Sol.

A: A1 will find a taxi waiting or will be picked up by bus after 5 min.

$$P(A) = \frac{2}{3} + \frac{1}{3}\frac{1}{2} = \frac{5}{6}$$

PMF

$$P_y(y) = \begin{cases} \frac{2}{3P(A)}, & \text{if } y = 0 \\ \frac{1}{6P(A)}, & \text{if } y = 5 \end{cases} = \begin{cases} \frac{12}{15}, & \text{if } y = 0 \\ \frac{31}{15}, & \text{if } y = 5 \end{cases}$$

$$P_(0) = P(Y = 0|A) = \frac{P(Y = 0, A)}{P(A)} = \frac{2}{3P(A)}$$

PDF

$$f_Z(z) = \begin{cases} \frac{1}{5}, & 0 \le z \le 5 \\ 0, & \text{else} \end{cases}$$

CDF

$$F_x(x) = P(A)F_Y(x) + (1 - P(A))F_Z(x) = \begin{cases} 0, & x < 0 \\ \frac{5}{6}\frac{12}{15} + \frac{1}{6}\frac{x}{5}, & 0 \le x < 5 \\ 1, & \text{if } 5 \le x \end{cases}$$

$$\begin{aligned} E(x) &= P(A)E[Y] + (1 - P(A))E[z] \\ &= \frac{5}{6}\frac{3}{15} * 5 + \frac{1}{6}\frac{5}{2} = \frac{15}{12} \end{aligned}$$

7. (10%) A city's temperature is modeled as a normal random variable with mean and standard deviation both equal to 10 degrees Celsius. What is the probability that the temperature at a random time is less than or equal to 59 degrees Fahrenheit?

   HW 3.13 Sol.

$$P(Y \leq 59) = P(X < 15) = P(Z \leq \frac{15 - E[x]}{\sigma x} = P(z \leq 0.5) = \Phi(0.5)$$

8. (10%) An absent-minded professor schedules two student appointments for the same time. The first student arrives on time while the second is late for 5 minutes. Suppose the duration of an appointment is exponential with mean 30 minutes. What is the expected value of the time between the arrival of the first student and the departure of the second student?

   HW 3.20 Sol.

$$\begin{aligned} E[Time] &= (5 + 30)(1 - e^{\frac{-5}{30}}) + (35 + 30)e^{\frac{-5}{30}} \\ &= 35 + 30e^{\frac{-5}{30}} \end{aligned}$$

9. (10%) Alice looks for her term paper in her filing cabinet, which has several drawers. She knows she left it in drawer j with probability $p_j > 0$. The drawers are so messy that even if she selects the correct drawer, say drawer i, she can only finds the paper with probability d_i. Suppose she opens drawer i and fails to find the paper.

   (a) What is the probability that the paper is in drawer j for $j \neq i$?

   (b) What is the probability that the paper is in drawer i?

   HW 1.19 Sol.

   A: the event that Alice does not find her paper in drawer i if $i \neq j$, $A \cap B = B$

$$P(B|A) = \frac{P(A \cap B)}{P(A)} = \frac{P(B)}{P(A)} = \frac{p_j}{1 - p_i d_i}$$

if $i = j$

$$P(B|A) = \frac{P(A \cap B)}{P(A)} = \frac{P(B)P(A|B)}{P(A)} = \frac{p_i(1 - d_i)}{1 - p_i d_i}$$

10. (10%) A jar initially contains m white and n black balls. Two players take turns removing a ball from the jar, and the first player removing a white ball wins. Find a recursive formula to compute the probability that the starting player wins.

    HW 1.21 Sol.

    $p(m, k)$: the probability that the starting player wins

$$\begin{aligned} p(m,k) &= \frac{m}{m+k} + \frac{k}{m+k}(1 - p(m, k-1)) \\ &= 1 - \frac{k}{m+k} p(m, k-1) \end{aligned}$$

11. (10%) Alice and Bob want to choose between movie and opera with equal probability. However, they only have a biased coin. Design a method to decide movie or opera with probability using the biased coin, and show that the method is indeed fair.

    HW 1.33 Sol.

    A_k: be the event that a decision was made at the kth round

$$p(opera) = \sum_{k=1}^{\infty} p(opera|A_k)p(A_k) = \sum_{k=1}^{\infty} \frac{1}{2} p(A_k) = \frac{1}{2}$$

$$\sum_{k=0}^{\infty} p(A_k) = 1$$

12. (10%) Suppose the probability of head of a coin toss is p. Let q_n be the probability of an even-number of heads in n tosses. Derive a recursive formula relating q_n and q_{n-1} and find q_n in terms of p and n.

HW 1.40 Sol.

A: event that the first n-1 tosses produce an even number of heads E: be the n-th toss is a head

$$
\begin{aligned}
q_n &= P((A \cap E^c) \cup (A^c \cap E)) = P(A \cap E^c) + P(A^c \cap E) \\
&= P(A)P(E^c) + P(A^c)P(E) = (1-p)q_{n-1} + p(1 - q_n - 2)
\end{aligned}
$$

$$
q_{n-1} = \frac{1 + (1-2p)^{n-1}}{2}
$$

$$
\begin{aligned}
q_n &= [(1 - q_{n-1}) + (1-p)q_{n-1} \\
&= p + (1-2p)q_{n-1} \\
&= p + (1-2p)\frac{1 + (1-2p)^{n-1}}{2} = \frac{1 + (1-2p)^n}{2}
\end{aligned}
$$